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TINGLEY'S PROBLEM FOR POSITIVE UNIT SPHERES OF OPERATOR ALGEBRAS AND DIAMETRAL RELATIONS

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ABSTRACT. We answer, in the affirmative, Tingley's problem for positive unit spheres of (complex) von Neumann algebras. More precisely, let $\Lambda : \mathrm{S}_{\mathcal{A}+} \to \mathrm{S}_{\mathcal{B}+}$ be a bijection between the sets of positive norm-one elements of two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$. We show that if Λ is an isometry, then it extends to a bijective complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. In the case in which Λ satisfies the weaker assumption of preserving pairs of points at diametrical distance, namely,

$$\|\Lambda(a) - \Lambda(b)\| = 1 \quad \text{if and only if} \quad \|a - b\| = 1 \quad (a, b \in \mathrm{S}_{\mathcal{A}+}),$$

one can still conclude that $\mathcal{A}$ is complex linear Jordan *-isomorphic to $\mathcal{B}$.

On our way, we also show that if there is an order isomorphism $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ between the projection lattices of $\mathcal{A}$ and $\mathcal{B}$ that preserves pairs of points at diametrical distance, that is,

$$\|\Theta(p) - \Theta(q)\| = 1 \quad \text{if and only if} \quad \|p - q\| = 1 \quad (p, q \in \mathrm{P}_{\mathcal{A}}),$$

then $\mathcal{A}$ and $\mathcal{B}$ are complex linear Jordan *-isomorphic. If, in addition, either $\mathcal{A}$ has no type $\mathbf{I}_2$ summand, or Θ is an isometry, then Θ extends to a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Actually, the above results are proved in the slightly more general situation that $\mathcal{A}$ and $\mathcal{B}$ are AW^*-algebras.

1. Introduction

An interesting question concerning the Banach space aspect of von Neumann algebras is to find a small metric subspace of a von Neumann algebra $\mathcal{A}$ that determines $\mathcal{A}$.

The seminal theorem of Kadison ([16,17]) tells us that every surjective real linear isometry between the self-adjoint parts $\mathcal{A}_{\mathrm{sa}}$ and $\mathcal{B}_{\mathrm{sa}}$ of two (complex) von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, induces a complex linear Jordan *-isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Furthermore, using the extension of the Mazur-Ulam Theorem ([23]) obtained by Mankiewicz ([22]), one knows that every surjective isometry between the closed unit balls, $\mathrm{B}_{\mathcal{A}_{\mathrm{sa}}}$ and $\mathrm{B}_{\mathcal{B}_{\mathrm{sa}}}$, of the real Banach spaces $\mathcal{A}_{\mathrm{sa}}$ and $\mathcal{B}_{\mathrm{sa}}$ extends to a real linear surjective isometry between $\mathcal{A}_{\mathrm{sa}}$ and $\mathcal{B}_{\mathrm{sa}}$. Consequently, the metric space $\mathrm{B}_{\mathcal{A}_{\mathrm{sa}}}$ determines $\mathcal{A}$ up to a complex linear Jordan *-isomorphism.

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Recently, Fernández-Polo and Peralta ([11]; see also [29]) showed that every surjective isometry between the unit spheres $\mathrm{S}_{\mathcal{A}}$ and $\mathrm{S}_{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$ induces a real linear Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$. The complexification of $J|_{\mathcal{A}_{\mathrm{sa}}} : \mathcal{A}_{\mathrm{sa}} \to \mathcal{B}_{\mathrm{sa}}$ is then a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. On the other hand, it was noted in [19, Proposition 3.2] that every surjective isometry between the positive closed unit balls $\mathrm{B}_{\mathcal{A}^+} = \mathrm{B}_{\mathcal{A}} \cap \mathcal{A}^+$ and $\mathrm{B}_{\mathcal{B}^+} = \mathrm{B}_{\mathcal{B}} \cap \mathcal{B}^+$ of $\mathcal{A}$ and $\mathcal{B}$ induces a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. In other words, the unit sphere $\mathrm{S}_{\mathcal{A}}$, as well as the positive closed unit ball $\mathrm{B}_{\mathcal{A}^+}$, also determines $\mathcal{A}$ up to a complex linear Jordan *-isomorphism.

These results motivate us to study the following problem (notice that the positive unit sphere $\mathrm{S}_{\mathcal{A}^+} = \mathrm{S}_{\mathcal{A}} \cap \mathcal{A}^+$ of $\mathcal{A}$ is smaller than all metric subspaces $\mathrm{B}_{\mathcal{A}_{\mathrm{sa}}}$, $\mathrm{S}_{\mathcal{A}}$ and $\mathrm{B}_{\mathcal{A}^+}$ in the above).

Problem 1.1. Does the metric space structure of the positive unit sphere $\mathrm{S}_{\mathcal{A}^+}$ of a complex von Neumann algebra $\mathcal{A}$ determine $\mathcal{A}$ up to a complex linear Jordan *-isomorphism?

This problem is related to the so-called "Tingley's problem for positive unit spheres". Recall that Tingley's problem asks whether a surjective isometry between the unit spheres of two real Banach spaces E and F extends to a surjective isometry between E and F ([39]). This problem has been studied extensively by many authors; see, e.g., [4,6,9,10,15,26,29,36,37]. Recently, certain order type analogues of Tingley's problem were studied in [20,21,26,31,36]. A question concerning this analogue was asked in [36], which was later generalized in [21] to the following general problem (see also [24–26,30]).

Problem 1.2 (Tingley's problem for positive unit spheres)**.** For a real ordered Banach space E with a generating cone E^+, denote by $\mathrm{S}_{E^+} := \{x \in E^+ : \|x\| = 1\}$ the positive unit sphere of E. Let $\Phi : \mathrm{S}_{E^+} \to \mathrm{S}_{F^+}$ be a surjective isometry between the positive unit spheres of two real ordered Banach spaces E and F with generating cones E^+ and F^+, respectively. Is there a bijective linear isometry $\check{\Phi} : E \to F$ extending Φ?

Problem 1.2 has affirmative answers when E and F are the self-adjoint parts of two matrix algebras ([31]), the self-adjoint parts of two algebras of bounded (respectively, compact) linear operators on (respectively, separable) Hilbert spaces ([37, Theorems 3.6 and 4.5]), as well as the self-adjoint parts of two commutative unital C^*-algebras ([20, Theorem 15]). Moreover, it also has an affirmative answer when E and F are the self-adjoint parts of type **I** finite von Neumann algebras that have bounded dimensions of irreducible representations ([21, Theorem 4.5]). Furthermore, an affirmative answer for Problem 1.2 when E and F are the self-adjoint parts of *predual* spaces of two von Neumann algebras was given in [26].

In this article, we will give an affirmative answer for Problem 1.2 in the case when E and F are the self-adjoint parts $\mathcal{A}_{\mathrm{sa}}$ and $\mathcal{B}_{\mathrm{sa}}$ of two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. This extends the corresponding results in [21,31,37], and gives a positive answer for Problem 1.1. In fact, if instead of assuming that the bijection is an isometry, we suppose that it enjoys the weaker hypothesis of preserving pairs of points at diametrical distance, we can get some meaningful conclusions.

Definition 1.3 ([33])**.** For a unital complex C^*-algebra A, the set of all pairs of elements in S_{A^+} at diametrical distance will be denoted by

$$\mathcal{D}_{\mathrm{S}_{A^+}} := \{(a,b) \in \mathrm{S}_{A^+} \times \mathrm{S}_{A^+} : \|a-b\| = 1\}, \tag{1.1}$$

which is called the *diametral relation* on the positive unit sphere S_{A^+}.

Obviously, for a unital complex C^*-algebra A, the diameter of S_{A^+} is 1 when $A \neq \mathbb{C}$ (note, however, that the diameter of $\mathrm{S}_{\mathbb{C}^+} = \{1\}$ is 0 and $\mathcal{D}_{\mathrm{S}_{\mathbb{C}^+}} = \emptyset$). In this case, $\mathcal{D}_{\mathrm{S}_{A^+}}$ is the set determines the points which are related by the relationship of being at diametrical distance.

In Sections 2 and 4, we will see that the diametral relation $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$ on the positive unit sphere $\mathrm{S}_{\mathcal{A}^+}$ of a von Neumann algebra $\mathcal{A}$ can tell us which elements in $\mathrm{S}_{\mathcal{A}^+}$ are projections (Lemma 4.1), which projections are central (Lemma 2.2(a)), and which central projection is the identity (Lemma 2.2(b)). It also determines the ordering on the set of projections (Lemma 4.2). Hence, the diametral relation $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$ recovers completely the projection lattice $\mathrm{P}_{\mathcal{A}}$ of $\mathcal{A}$. Furthermore, $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$ determines the Jordan *-structure of $\mathcal{A}$.

The following result will be proved in Proposition 4.5 and Theorem 4.8. Part (c) of it gives an affirmative answer to Problem 1.2 in the case that E and F are the self-adjoint parts of two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, respectively.

Theorem 1.4. *Suppose that there is a bijection $\Lambda : \mathrm{S}_{\mathcal{A}^+} \to \mathrm{S}_{\mathcal{B}^+}$ between the positive unit spheres of two (complex) von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ preserving pairs of points at diametrical distance; that is,*

$$\text{for } a,b \in \mathrm{S}_{\mathcal{A}^+}, \text{ one has } \|a-b\| = 1 \text{ if and only if } \|\Lambda(a)-\Lambda(b)\| = 1.$$

Then the following statements hold:

(a) *$\mathcal{A}$ is (complex linear) Jordan *-isomorphic to $\mathcal{B}$.*
(b) *When $\mathcal{A}$ has no type $\mathbf{I}_2$ summand, the restriction $\Lambda|_{\mathrm{P}_{\mathcal{A}}\setminus\{0\}}$ extends to a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*
(c) *If Λ is an isometry, then it extends to a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

As an essential step toward the above theorem, we will give a general result concerning extensions of a bijective map $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ between projection lattices of two von Neumann algebras. The first of such extension results was obtained by Dye ([8]), and states that Θ extends to a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ whenever $\mathcal{A}$ has no type $\mathbf{I}_2$ summand and Θ is *bi-orthogonality preserving*, in the sense that

$$\Theta(p)\Theta(q) = 0 \quad \text{if and only if} \quad pq = 0 \quad (p,q \in \mathrm{P}_{\mathcal{A}}).$$

However, it is well-known that this conclusion will not hold if one weakens the bi-orthogonality preserving assumption to that of Θ being a lattice isomorphism (i.e., an order isomorphism).

In fact, supplementing the work of von Neumann in the type $\mathbf{II}_1$ case ([32]), a complete description of lattice isomorphisms between projection lattices of two von Neumann algebras was given in [28]. On the other hand, it was shown in [27] that if a bijection $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ is an isometry instead, then $\mathcal{A}$ is complex linear Jordan *-isomorphic to $\mathcal{B}$, provided that $\mathcal{A}$ has no type $\mathbf{I}_1$ summand (see also [12,13]). However, in this case, the bijective metric isometry Θ need not extend to a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Nevertheless, it was shown in [21, Proposition 4.4] that if $\mathcal{A}$ and $\mathcal{B}$ are type $\mathbf{I}$ finite von Neumann algebras, and Θ is an isometric lattice isomorphism, then Θ extends to a Jordan *-isomorphism. In order to prove Theorem 1.4, we will first show that [21, Proposition 4.4] can be extended to general von Neumann algebras. We will also consider the weaker hypothesis that the order isomorphism Θ preserves only the diametral relations. Indeed, in a similar fashion as above, we define the diametral relation on the projection lattice $\mathrm{P}_{\mathcal{A}}$ of a von Neumann algebra $\mathcal{A}$ by

$$\mathcal{D}_{\mathrm{P}_{\mathcal{A}}} := \big\{(p,q) \in \mathrm{P}_{\mathcal{A}} \times \mathrm{P}_{\mathcal{A}} : \|p-q\| = 1\big\}. \tag{1.2}$$

Note that the diameter of $\mathrm{P}_{\mathcal{A}}$ is always 1.

The following result, which is established in Proposition 3.2 and Theorem 3.4, affirms that the metric space structure, or even the weaker structure of all pairs of points at diametrical distance, *together with* the ordered space structure of the projection lattice $\mathrm{P}_{\mathcal{A}}$ of a von Neumann algebra $\mathcal{A}$ determine $\mathcal{A}$ up to a complex linear Jordan *-isomorphism.

Theorem 1.5. *Suppose that there is an order isomorphism $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ between the projection lattices of two complex von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ preserving pairs of projections at diametrical distance; i.e., for $p, q \in \mathrm{P}_{\mathcal{A}}$, one has $\|p-q\| = 1$ if and only if $\|\Theta(p) - \Theta(q)\| = 1$. Then the following statements hold:*

(a) *$\mathcal{A}$ is (complex linear) Jordan *-isomorphic to $\mathcal{B}$.*

(b) *When $\mathcal{A}$ has no type $\mathbf{I}_2$ summand or when Θ is an isometry, Θ extends to a complex linear Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

Note that the above conclusion is not at all trivial, since neither a lattice isomorphism nor an isometric bijection between the projection lattices of two von Neumann algebras is necessarily bi-orthogonality preserving, and thus we cannot apply directly the results in [8]. Nevertheless, we will show that when an order isomorphism preserves the diametral relations as well, it is indeed bi-orthogonality preserving.

Moreover, we will see in Corollaries 4.9 and 3.6 that the conclusions in Theorems 1.4 and 1.5 are valid if one replaces complex linear Jordan *-isomorphisms and complex linear Jordan *-isomorphic with real linear *-isomorphisms and real linear *-isomorphic, respectively.

Although all our results mentioned in the above are stated for complex von Neumann algebras, we actually establish Theorems 1.4 and 1.5, and thus solve Problem 1.1, in the more general case that $\mathcal{A}$ and $\mathcal{B}$ are complex AW^*-algebras.

For information concerning AW^*-algebras, the readers may consult, e.g., [1, 14, 18].

2. Notations and some elementary results

From now on, all C^*-algebras (in particular, all AW^*-algebras) are over the complex field $\mathbb{C}$, and all Jordan *-isomorphisms are assumed to be complex linear, unless specified.

Let A be a C^*-algebra. The subsets

$$\mathrm{B}_{A^+} := \{a \in A^+ : \|a\| \leq 1\} \quad \text{and} \quad \mathrm{S}_{A^+} := \{a \in A^+ : \|a\| = 1\},$$

are called the *positive unit ball* and the *positive unit sphere* of A, respectively. Denote by $\mathrm{Z}(A)$ the center of A, and by P_A the lattice of all projections in A.

When A is unital, we write A^{-1} for the set of invertible elements of A, and put

$$\mathrm{B}_{A^+}^{-1} := \mathrm{B}_{A^+} \cap A^{-1} \quad \text{as well as} \quad \mathrm{S}_{A^+}^{-1} := \mathrm{S}_{A^+} \cap A^{-1}.$$

Moreover, $\mathcal{PS}(A)$ denotes the set of all pure states on A.

We list some known facts in Lemma 2.1 (see, e.g., [21]).

Lemma 2.1. *Let A be a C^*-algebra, $p, q \in \mathrm{P}_A$ and $a, b \in \mathrm{S}_{A^+}$.*

(a) $\|a - b\| = 1$ *if and only if there is* $\omega \in \mathcal{PS}(A)$ *with* $\{\omega(a), \omega(b)\} = \{0, 1\}$.

(b) *Suppose that A is unital. Then $a \in A^{-1}$ (i.e., $a \in \mathrm{S}_{A^+}^{-1}$) if and only if $\omega(a) \neq 0$ for all $\omega \in \mathcal{PS}(A)$, which is also equivalent to $\|1 - a\| < 1$.*

(c) *If $\omega \in \mathcal{PS}(A)$ with $\omega(p) = 1$, then $\omega(pap) = \omega(a)$.*

(d) *If $c \in A^+$ satisfying $cq = 0$, $\|c\| < 1$ and $p \leq q + c$, then $p \leq q$.*

(e) *When A is unital, one has $\left\{c \in \mathrm{S}_{A^+}^{-1} : p \leq c\right\} = p + \mathrm{B}_{((1-p)A(1-p))^+}^{-1}$.*

(f) *$pbp = 0$ if and only if $pb = 0$.*

For a C^*-algebra A and $E \subseteq \mathrm{S}_{A^+}$, we set

$$E^{<1} := \left\{a \in \mathrm{S}_{A^+} : \|a - e\| < 1, \text{ for every } e \in E\right\};$$

$$E^{1} := \left\{a \in \mathrm{S}_{A^+} : \|a - e\| = 1, \text{ for every } e \in E\right\}. \tag{2.1}$$

One may consider $(\mathrm{S}_{\mathcal{A}^+}, \mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}})$ as an ortho-set (in the sense of [7]). In this case, E^1 is the "ortho-complementation" of E as in [7]. Notice also that

$$\{e\}^{<1} = \mathrm{S}_{\mathcal{A}^+} \setminus \{e\}^1 \quad \text{for any } e \in \mathrm{S}_{A^+}. \tag{2.2}$$

However, it is possible that $E^{<1} \subsetneq \mathrm{S}_{\mathcal{A}^+} \setminus E^1$ when E is not a singleton set.

Our next lemma ensures that one can determine whether a projection is central using the diametral relation $\mathcal{D}_{\mathrm{P}_A}$ on P_A (see (1.2)), and that one can determine whether a non-zero central projection is the identity element using the diametral relation $\mathcal{D}_{\mathrm{S}_{A^+}}$ on S_{A^+} (see (1.1)).

Lemma 2.2. *Let A be a unital C^*-algebra.*

(a) *A projection $p \in \mathrm{P}_A$ is central if and only if $\|p - q\| = 1$ for every $q \in \mathrm{P}_A \setminus \{p\}$.*

(b) *For a non-zero central projection $p \in \mathrm{P}_{\mathrm{Z}(A)} \setminus \{0\}$, the following are equivalent.*

(1) *p is the identity 1 of A.*

(2) *$E^{<1} \cap \{p\}^{<1} \neq \emptyset$, whenever E is a non-empty subset of S_{A^+} with $E^{<1} \neq \emptyset$.*

(3) *$\{a\}^1 \cup \{p\}^1 \neq \mathrm{S}_{A^+}$, for every $a \in \mathrm{S}_{A^+}$.*

Proof.

(a) This part follows directly from [5, Proposition 2.1].

(b) Suppose first that $p = 1$. Let $\emptyset \neq E \subseteq \mathrm{S}_{A^+}$ such that there is an element e in $E^{<1}$. Set

$$c := (1 + e)/2.$$

Clearly, $\|1 - c\| = \|1 - e\|/2 \leq 1/2$, and hence $c \in \{1\}^{<1} = \{p\}^{<1}$. On the other hand, for any $b \in E$, as $\|e - b\| < 1$, we know that

$$\|c - b\| = \|(1 - b)/2 + (e - b)/2\| \leq 1/2 + \|e - b\|/2 < 1.$$

Hence, $c \in E^{<1}$ as well. Thus, Condition (2) holds.

Secondly, if Condition (2) holds, then $\{a\}^{<1} \cap \{p\}^{<1} \neq \emptyset$ $(a \in \mathrm{S}_{A^+})$, and Condition (3) follows from (2.2).

Finally, suppose that Condition (3) holds, but $p \neq 1$; i.e., $1 - p \in \mathrm{S}_{A^+}$. Let $b \in \mathrm{S}_{A^+}$. Since p is central, one has $b = bp + b(1-p) = pbp + (1-p)b(1-p)$, and hence

$$1 = \|b\| = \|bp + b(1-p)\| = \max\big\{\|bp\|, \|b(1-p)\|\big\}.$$

If $\|bp\| = 1$, then

$$\|(1-p) - b\| = \|(1-b)(1-p) - bp\| = \max\big\{\|(1-b)(1-p)\|, \|bp\|\big\} = 1,$$

and thus $b \in \{1-p\}^1$. On the other hand, if $\|b(1-p)\| = 1$, then $\|p - b\| = \|(1-b)p - b(1-p)\| = 1$, and thus $b \in \{p\}^1$. The above conclusions lead to the contradiction that $\{p\}^1 \cup \{1-p\}^1 = \mathrm{S}_{A^+}$. □

3. Isometric order isomorphisms between projection lattices of AW^*-algebras

In this section, we will establish Theorem 1.5 in the context of AW^*-algebras. Let us begin with the following technical lemma, which tells us how to determine whether two projections are orthogonal, via the diametral relation $\mathcal{D}_{\mathrm{P}_\mathcal{A}}$ (see (1.2)) and the order relation on $\mathrm{P}_\mathcal{A}$.

Lemma 3.1. *Let $\mathcal{A}$ be an AW^*-algebra, and $p, q \in \mathrm{P}_\mathcal{A} \setminus \{0\}$. Then $pq = 0$ if and only if $\|r - s\| = 1$ for every $r, s \in \mathrm{P}_\mathcal{A} \setminus \{0\}$ with $r \leq p$ and $s \leq q$.*

Proof. The necessity is obvious, and we will only check the sufficiency. By [14, Proposition 2.5] (which is the AW^*-analogue of a result of Pedersen; see [34, Theorem 3.4]), there are Stonean spaces X and Y (in fact, X can be chosen to be a finite set) as well as an AW^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that

$$p, q \in \mathcal{B} \quad \text{and} \quad \mathcal{B} \cong C(X) \oplus C(Y; \mathbb{M}_2),$$

where $\mathbb{M}_2$ is the algebra of 2×2 matrices. Using this identification, one can find $p_1, q_1 \in \mathrm{P}_{C(X)}$ and $p_2, q_2 \in \mathrm{P}_{C(Y;\mathbb{M}_2)}$ such that

$$p = p_1 + p_2 \quad \text{and} \quad q = q_1 + q_2.$$

We will use the canonical identification $C(Y; \mathbb{M}_2) \cong C(Y) \otimes \mathbb{M}_2$, and write $\mathbf{1}_O$ for the indicator function of a clopen subset $O \subseteq Y$. Consider the clopen sets

$$U_\alpha := \{y \in Y : p_2(y) \text{ is of rank one}\}, \quad U_\beta := \{y \in Y : p_2(y) = 1_{\mathbb{M}_2}\},$$
$$V_\alpha := \{y \in Y : q_2(y) \text{ is of rank one}\}, \quad V_\beta := \{y \in Y : q_2(y) = 1_{\mathbb{M}_2}\}.$$

Assume on the contrary that the said condition in the statement holds, but $pq \neq 0$. If $p_1 q_1 \neq 0$, then $r_1 := p_1 q_1 \in \mathrm{P}_{C(X)} \setminus \{0\}$ will satisfy $r_1 \leq p_1 \leq p$ as well as $r_1 \leq q_1 \leq q$. However, by the assumed condition, we have $\|r_1 - r_1\| = 1$, which is impossible. Consequently, we know that $p_2 q_2 \neq 0$, and so, there is $y_0 \in Y$ such that $p_2(y_0) q_2(y_0) \neq 0$. One has

$$y_0 \in \big(U_\alpha \cap V_\alpha\big) \cup \big(U_\alpha \cap V_\beta\big) \cup \big(U_\beta \cap V_\alpha\big) \cup \big(U_\beta \cap V_\beta\big).$$

If $U_\alpha \cap V_\beta \neq \emptyset$, then $r_2 := (\mathbf{1}_{U_\alpha \cap V_\beta} \otimes 1_{\mathbb{M}_2}) p_2$ is a non-zero projection with

$$r_2 \leq p_2 \leq p \quad \text{and} \quad r_2 \leq q_2 \leq q.$$

By the assumed condition, we have $\|r_2 - r_2\| = 1$, which is impossible. A similar contradiction will exist if either $U_\beta \cap V_\alpha \neq \emptyset$ or $U_\beta \cap V_\beta \neq \emptyset$. Therefore, we are left with the only possibility that $y_0 \in U_\alpha \cap V_\alpha$. This means that both $p_2(y_0)$ and $q_2(y_0)$ are rank one projections in $\mathbb{M}_2$ and they are not orthogonal. In this case,

$\|p_2(y_0) - q_2(y_0)\| < \kappa$ for some $\kappa \in (0,1)$. By the continuity of p_2 and q_2, there is a clopen subset $W \subseteq Y$ such that $\|p_2(z) - q_2(z)\| \leq \kappa$ $(z \in W)$. Now, the projections

$$r := (\mathbf{1}_W \otimes 1_{\mathbb{M}_2})p_2 \quad \text{and} \quad s := (\mathbf{1}_W \otimes 1_{\mathbb{M}_2})q_2$$

in $\mathrm{P}_{\mathcal{B}} \setminus \{0\}$ will satisfy $r \leq p_2 \leq p$, $s \leq q_2 \leq q$, but $\|r - s\| \leq \kappa < 1$, which conflicts with the assumed condition. □

Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be AW^*-algebras. Suppose that there is an order isomorphism $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ that preserves pairs of points at diametrical distance, i.e., $(\Phi \times \Phi)(\mathcal{D}_{\mathrm{P}_{\mathcal{A}}}) = \mathcal{D}_{\mathrm{P}_{\mathcal{B}}}$. Then the following statements hold:*

(a) *Θ is bi-orthogonality preserving and $\Theta(\mathrm{P}_{\mathrm{Z}(\mathcal{A})}) = \mathrm{P}_{\mathrm{Z}(\mathcal{B})}$.*
(b) *If $\mathcal{A}$ has no type $\mathbf{I}_2$ summand, then Θ extends to a Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*
(c) *Let $e_0 \in \mathrm{P}_{\mathrm{Z}(\mathcal{A})}$ and $f_0 \in \mathrm{P}_{\mathrm{Z}(\mathcal{B})}$ be the central projections for the type $\mathbf{I}_2$ parts of $\mathcal{A}$ and $\mathcal{B}$, respectively. If we set $e_1 := 1 - e_0$ and $f_1 := 1 - f_0$, then $\Theta(e_0\mathrm{P}_{\mathcal{A}}) = f_0\mathrm{P}_{\mathcal{B}}$ and $\Theta(e_1\mathrm{P}_{\mathcal{A}}) = f_1\mathrm{P}_{\mathcal{B}}$.*
(d) *$\mathcal{A}$ is Jordan *-isomorphic to $\mathcal{B}$.*

Proof.

(a) It follows from Lemma 3.1 that Θ is bi-orthogonality preserving. Moreover, the equality $\Theta(\mathrm{P}_{\mathrm{Z}(\mathcal{A})}) = \mathrm{P}_{\mathrm{Z}(\mathcal{B})}$ follows from Lemma 2.2(a).

(b) This follows from part (a) above and [14, Theorem 4.3].

(c) Since Θ is an order isomorphism and $\Theta(\mathrm{P}_{\mathrm{Z}(\mathcal{A})}) = \mathrm{P}_{\mathrm{Z}(\mathcal{B})}$ (see part (a) above), we know that $\Theta(e_1) \in \mathrm{P}_{\mathrm{Z}(\mathcal{B})}$ and that Θ sends

$$\mathrm{P}_{e_1\mathcal{A}} = e_1\mathrm{P}_{\mathcal{A}} = \{p \in \mathrm{P}_{\mathcal{A}} : p \leq e_1\}$$

onto $\mathrm{P}_{\Theta(e_1)\mathcal{B}} = \Theta(e_1)\mathrm{P}_{\mathcal{B}}$. Part (b) above then implies that $\Theta(e_1)\mathcal{B}$ does not have a type $\mathbf{I}_2$ summand, which gives $\Theta(e_1) \leq f_1$. By considering Θ^{-1}, we know that $\Theta(e_1) = f_1$. Hence, $\Theta(e_0) = f_0$ as well. This gives the required equalities.

(d) It follows from parts (b) and (c) above that $e_1\mathcal{A}$ is Jordan *-isomorphic to $f_1\mathcal{B}$. On the other hand, we learn from parts (a) and (c) above that $\Theta(\mathrm{P}_{\mathrm{Z}(e_0\mathcal{A})}) = \mathrm{P}_{\mathrm{Z}(f_0\mathcal{B})}$. Part (b) above then ensures that the commutative AW^*-algebras $\mathrm{Z}(e_0\mathcal{A})$ is *-isomorphic to $\mathrm{Z}(f_0\mathcal{B})$. Consequently, $e_0\mathcal{A}$ is *-isomorphic to $f_0\mathcal{B}$ (because a type $\mathbf{I}_2$ AW^*-algebra assumes the form $C(X, \mathbb{M}_2)$ for a Stonean space X with center $C(X)$; see [18, Theorem 1])). □

Note that part (b) above may fail when $\mathcal{A}$ has a type $\mathbf{I}_2$ summand. The following is such an example.

Example 3.3. Let $e_0 \in \mathrm{P}_{\mathbb{M}_2}$ be a rank one projection. Define $\Theta : \mathrm{P}_{\mathbb{M}_2} \to \mathrm{P}_{\mathbb{M}_2}$ by

$$\Theta(p) := p \qquad (p \in \mathrm{P}_{\mathbb{M}_2} \setminus \{e_0, 1 - e_0\}),$$

$\Theta(e_0) := 1 - e_0$ and $\Theta(1 - e_0) := e_0$. Then Θ is a *discontinuous* bi-orthogonality preserving bijection. Clearly, Θ is an order isomorphism. Moreover, it is obvious that Θ preserves pairs of points at diametrical distance. However, the discontinuous map Θ cannot be extended to a linear map from $(\mathbb{M}_2)_{\mathrm{sa}}$ to itself.

Theorem 3.4 below extends [21, Proposition 4.5] to general AW^*-algebras (while [21, Proposition 4.5] only proved the corresponding statement for type $\mathbf{I}$ finite von Neumann algebras).

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be AW^*-algebras. If $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ is an isometric order isomorphism, then Θ extends to a Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. Let e_0 and f_0 be the central projections in $\mathcal{A}$ and $\mathcal{B}$ such that $e_0\mathcal{A}$ and $f_0\mathcal{B}$ are the type $\mathbf{I}_2$ parts of $\mathcal{A}$ and $\mathcal{B}$, respectively. Proposition 3.2(a) tells us that

$$\Theta(p) = \Theta(e_0 p) \vee \Theta((1-e_0)p) = \Theta(e_0 p) + \Theta((1-e_0)p) \qquad (p \in \mathrm{P}_{\mathcal{A}}).$$

As in the proof of Proposition 3.2(c), the map $e_0 p \mapsto \Theta(e_0 p)$ will send $\mathrm{P}_{e_0\mathcal{A}}$ onto $\mathrm{P}_{f_0\mathcal{B}}$ and the map $(1-e_0)p \mapsto \Theta((1-e_0)p)$ will send $\mathrm{P}_{(1-e_0)\mathcal{A}}$ onto $\mathrm{P}_{(1-f_0)\mathcal{B}}$. Note that both of these two maps are isometric order isomorphisms. Hence, Proposition 3.2(b) shows that the map $(1-e_0)p \mapsto \Theta((1-e_0)p)$ extends to a Jordan *-isomorphism between the AW^*-algebras $(1-e_0)\mathcal{A}$ and $(1-f_0)\mathcal{B}$. We are thus left with the type $\mathbf{I}_2$ case.

Therefore, we assume, without loss of generality, that both $\mathcal{A}$ and $\mathcal{B}$ are type $\mathbf{I}_2$ AW^*-algebras. Let X and Y be Stonean spaces such that $\mathcal{A} \cong C(X, \mathbb{M}_2)$ and $\mathcal{B} \cong C(Y, \mathbb{M}_2)$. Proposition 3.2(a) tells us that $\Theta(\mathrm{P}_{\mathrm{Z}(\mathcal{A})}) = \mathrm{P}_{\mathrm{Z}(\mathcal{B})}$, and Proposition 3.2(b) implies that $\Theta|_{\mathrm{P}_{\mathrm{Z}(\mathcal{A})}}$ extends to a *-isomorphism from $\mathrm{Z}(\mathcal{A}) \cong C(X)$ onto $\mathrm{Z}(\mathcal{B}) \cong C(Y)$. This gives a homeomorphism $\sigma : X \to Y$ satisfying $\Theta(\mathbf{1}_V \otimes 1_{\mathbb{M}_2}) = \mathbf{1}_{\sigma(V)} \otimes 1_{\mathbb{M}_2}$ for any clopen subset $V \subseteq X$. Thus, for $p \in \mathrm{P}_{\mathcal{A}}$, we have

$$\begin{aligned} \Theta\big((\mathbf{1}_V \otimes 1_{\mathbb{M}_2})p\big) &= \Theta\big((\mathbf{1}_V \otimes 1_{\mathbb{M}_2}) \wedge p\big) \\ &= \Theta(\mathbf{1}_V \otimes 1_{\mathbb{M}_2}) \wedge \Theta(p) = (\mathbf{1}_{\sigma(V)} \otimes 1_{\mathbb{M}_2})\Theta(p). \end{aligned} \tag{3.1}$$

Fix $x \in X$. We first claim that one can find a map $\Phi_x : \mathrm{P}_{\mathbb{M}_2} \to \mathrm{P}_{\mathbb{M}_2}$ satisfying

$$\Phi_x(p(x)) = \Theta(p)(\sigma(x)) \qquad (p \in \mathrm{P}_{\mathcal{A}}). \tag{3.2}$$

In fact, consider $e \in \mathrm{P}_{\mathbb{M}_2}$. There exists $p \in \mathrm{P}_{\mathcal{A}}$ with $p(x) = e$, and we put

$$\Phi_x(e) := \Theta(p)(\sigma(x)) \in \mathrm{P}_{\mathbb{M}_2}.$$

In order to show that Φ_x is well-defined, suppose that $q \in \mathrm{P}_{\mathcal{M}}$ also satisfies $q(x) = e$. Let $\epsilon > 0$ be arbitrarily small. There is a clopen subset $V_1 \subseteq X$ containing x with

$$\|p(y) - e\| < \epsilon/2 \quad \text{and} \quad \|q(y) - e\| < \epsilon/2 \qquad (y \in V_1).$$

Thus, $\|(\mathbf{1}_{V_1} \otimes 1_{\mathbb{M}_2})p - (\mathbf{1}_{V_1} \otimes 1_{\mathbb{M}_2})q\| \leq \epsilon$. The hypothesis and Relation (3.1) imply

$$\big\|\Theta(p)(\sigma(x)) - \Theta(q)(\sigma(x))\big\| \leq \epsilon.$$

Hence, $\Theta(p)(\sigma(x)) = \Theta(q)(\sigma(x))$, and Φ_x is a well-defined map satisfying Relation (3.2).

Secondly, we claim that Φ_x is a surjective isometry, with $\Phi_x(0_{\mathbb{M}_2}) = 0_{\mathbb{M}_2}$ and $\Phi_x(1_{\mathbb{M}_2})$
$= 1_{\mathbb{M}_2}$. In fact, we let $e_1, e_2 \in \mathrm{P}_{\mathbb{M}_2}$, and set

$$p_1 := 1_{C(X)} \otimes e_1 \quad \text{and} \quad p_2 := 1_{C(X)} \otimes e_2.$$

Then $\|e_1 - e_2\| = \|p_1 - p_2\|$. This gives

$$\|\Phi_x(e_1) - \Phi_x(e_2)\| = \|\Theta(p_1)(\sigma(x)) - \Theta(p_2)(\sigma(x))\| \leq \|\Theta(p_1) - \Theta(p_2)\| = \|e_1 - e_2\|.$$

Consequently, Φ_x is a contraction. By considering Θ^{-1}, one sees that Φ_x is a surjective isometry. Now, if $e_1 = 1_{\mathbb{M}_2}$, then $p_1 = 1_{\mathcal{A}}$, and the order preserving assumption of Θ gives $\Theta(p_1) = 1_{\mathcal{B}}$, which implies $\Phi_x(1_{\mathbb{M}_2}) = 1_{\mathbb{M}_2}$. Similarly, $\Phi_x(0_{\mathbb{M}_2}) = 0_{\mathbb{M}_2}$, and these establish our second claim.

We know, from the second claim above, that Φ_x restricts to a surjective isometry from $\mathrm{P}_{\mathbb{M}_2} \setminus \{0, 1\}$ onto itself. It then follows from Wigner's theorem (see, e.g., [2, Theorem 1.1]) that Φ_x extends to a Jordan *-isomorphism, again denoted by Φ_x, from $\mathbb{M}_2$ to $\mathbb{M}_2$.

Finally, the bijection $\sigma : X \to Y$ and the collection $\{\Phi_x\}_{x\in X}$ of Jordan *-isomorphisms induce a Jordan *-isomorphism $\Phi : \ell^\infty(X;\mathbb{M}_2) \to \ell^\infty(Y;\mathbb{M}_2)$ with

$$\Phi(g)(\sigma(x)) = \Phi_x(g(x)) \qquad (g \in \ell^\infty(X;\mathbb{M}_2)).$$

It follows from (3.2) that

$$\Phi(p) = \Theta(p) \qquad (p \in \mathrm{P}_{\mathcal{A}}).$$

As $\mathrm{P}_{\mathcal{A}}$ and $\mathrm{P}_{\mathcal{B}}$ generate the AW^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\ell^\infty(X;\mathbb{M}_2)$ and $\ell^\infty(Y;\mathbb{M}_2)$, respectively, the above produces the required equality $\Phi(\mathcal{A}) = \mathcal{B}$. □

In the particular case of von Neumann algebras, we can restate the conclusion of Theorem 3.4 such that Θ extends to a real linear *-isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$, because of the following well-known lemma.

Lemma 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras.*

(a) *If $\Psi : \mathcal{A} \to \mathcal{B}$ is a complex linear Jordan *-isomorphism, then there is a real linear *-isomorphism $\Gamma : \mathcal{A} \to \mathcal{B}$ with $\Gamma|_{\mathcal{A}_{\rm sa}} = \Psi|_{\mathcal{A}_{\rm sa}}$.*

(b) *If $\Theta : \mathcal{A} \to \mathcal{B}$ is a real linear *-isomorphism, then the complexification of $\Theta|_{\mathcal{A}_{\rm sa}}$ is a complex linear Jordan *-isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$.*

Proof.

(a) By [38, Theorem 3.3], there is a central projection q in $\mathcal{B}$ such that $a \mapsto q\Psi(a)$ is a *-homomorphism and $a \mapsto (1-q)\Psi(a)$ is a *-anti-homomorphism. If we define $\Gamma : \mathcal{A} \to \mathcal{B}$ by

$$\Gamma(a) := \Psi\big(pa + (1-p)a^*\big) \qquad (a \in \mathcal{A}),$$

where p is the central projection in $\mathcal{A}$ satisfying $\Psi(p) = q$, then Γ is a real linear *-isomorphism with $\Gamma|_{\mathcal{A}_{\rm sa}} = \Psi|_{\mathcal{A}_{\rm sa}}$.

(b) This follows from the fact that $\Theta|_{\mathcal{A}_{\rm sa}}$ is a Jordan isomorphism from $\mathcal{A}_{\rm sa}$ onto $\mathcal{B}_{\rm sa}$. □

Corollary 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras with projection lattices P_A and P_B, respectively. Any isometric order isomorphism $\Theta : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ extends to a real linear *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

It seems that Lemma 3.5 might also hold for complex AW^*-algebras, if one uses [3, Theorem 2.3] to replace [38, Theorem 3.3] (note, however, that some work is needed to go from the conclusion concerning essential ideal as in [3, Theorem 2.3] to the whole algebra).

4. Bijective isometries between positive unit spheres of AW^*-algebras

The aim of this section is to establish Theorem 1.4 in the context of AW^*-algebras. Let us begin with the following result, which tells us that one can determine whether an element $a \in \mathrm{S}_{\mathcal{A}^+}$ is a projection, via the diametral relation $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$ on $\mathrm{S}_{\mathcal{A}^+}$ (see Definition 1.3). This result generalizes both [35, Theorem 2.3] and [21, Lemma 3.2(a)]. However, neither the proof for atomic von Neumann algebras as in [35] nor the one for type **I** finite von Neumann algebras as in [21] works in this general case.

Lemma 4.1. *Suppose that $\mathcal{A}$ is an AW^*-algebra. Then $p \in \mathrm{S}_{\mathcal{A}^+}$ is a projection if and only if $\{p\} = (\{p\}^1)^1$, where E^1 is defined as in* (2.1).

Proof. The sufficiency was established in [35, Proposition 2.2] for general C^*-algebras. For the necessity, suppose that p is a non-zero projection and consider $a \in (\{p\}^1)^1$. We need to verify that $a = p$.

We first claim that

$$\|qaq\| = 1 \quad \text{whenever} \quad q \in \mathrm{P}_{\mathcal{A}} \setminus \{0\} \quad \text{satisfying} \quad q \leq p.$$

Assume on the contrary that $\|qaq\| < 1$. Suppose further that $(1-q)a(1-q) = 0$ as well. Then Lemma 2.1(f) gives $a(1-q) = 0$, i.e., $a = aq$. This implies

$$a = aq = qa = qaq,$$

and we will arrive at the contradiction that $\|qaq\| = \|a\| = 1$. Consequently, $(1-q)a(1-q) \neq 0$. In this case, there exist $r \in \mathrm{P}_{\mathcal{A}} \setminus \{0, 1\}$ and $\alpha \in (0, 1)$ with $\alpha r \leq (1-q)a(1-q)$. In particular, $r \leq 1-q$ (because $rq = 0$) and we have

$$\alpha r \leq rar. \tag{4.1}$$

Set $b := (1-q+r)/2 \in \mathrm{S}_{((1-q)\mathcal{A}(1-q))^+}$. We know from $p - q \in \mathrm{P}_{(1-q)\mathcal{A}(1-q)}$ that

$$\|p - b\| = \|q + (p-q) - b\| = \max\{\|q\|, \|(p-q) - b\|\} = 1.$$

As $a \in (\{p\}^1)^1$, one has $\|a - b\| = 1$. By Lemma 2.1(a), there exists $\omega \in \mathcal{PS}(\mathcal{A})$ satisfying

$$\{\omega(a), \omega(b)\} = \{0, 1\}.$$

If $\omega(b) = 1$, then it follows from $\omega(1-q) + \omega(r) = 2\omega(b) = 2$ that $\omega(r) = 1$ (as $\omega(1-q), \omega(r) \in [0,1]$), but this, together with Lemma 2.1(c) and (4.1), gives the contradiction that

$$\omega(a) = \omega(rar) \geq \alpha\omega(r) > 0.$$

If $\omega(b) = 0$, then $\omega(q) = 1$ (because $1 - q \leq 2b$), but this gives

$$\omega(a) = \omega(qaq) \leq \|qaq\| < 1,$$

which contradicts $\omega(a) = 1$. This establishes our first claim.

Secondly, we claim that

$$pap = p.$$

Suppose on the contrary that $\epsilon := \|p - pap\|/2 > 0$. Let $\mathcal{B}$ be the abelian AW^*-subalgebra of $p\mathcal{A}p$ generated by pap and p. Consider X to be the Stonean space with $C(X) \cong \mathcal{B}$, and $g \in C(X)_+$ to be the function corresponding to $p - pap \in \mathcal{B}_+$. Denote by $q_0 \in \mathrm{P}_{\mathcal{B}} \setminus \{0\} \subseteq \mathrm{P}_{p\mathcal{A}p} \setminus \{0\}$ the projection corresponding to the indicator function of the closure of the open subset $g^{-1}\big((\epsilon, +\infty)\big)$. Then

$$q_0 - q_0 a q_0 = q_0(p - pap)q_0 \geq \epsilon q_0.$$

From this, we see that $q_0 a q_0 \leq (1-\epsilon)q_0$, which implies that $\|q_0 a q_0\| \leq 1 - \epsilon$. However, this conflicts with the first claim above.

The second claim above implies that $p(1-a)p = 0$. This means that $p(1-a) = 0$, i.e., $p = pa$. Hence, $ap = pa = p$, and one has

$$a = pa + (1-p)a = p + (1-p)a(1-p).$$

If $d := (1-p)a(1-p) \neq 0$, then $p - d/\|d\| \in \{p\}^1$. Since $a \in (\{p\}^1)^1$ and $a = p + d$, we have the following contradiction:

$$1 = \big\|a - (p - d/\|d\|)\big\| = \big\|d + d/\|d\|\big\| = \|d\| + 1.$$

Consequently, $d = 0$, which gives $a = p$, as asserted. □

Through the discussion following (2.1), Lemma 4.1 can be restated as that $a \in \mathrm{S}_{\mathcal{A}^+}$ is a projection if and only if $\{a\}$ is an ortho-subset of the ortho-set $(\mathrm{S}_{\mathcal{A}^+}, \mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}})$ (see [7, Definition 2.1]).

Lemma 4.2. *Let $\mathcal{A}$ be an AW^*-algebra, and $p, q \in \mathrm{P}_\mathcal{A} \setminus \{0\}$. Then*

$$q \leq p \quad \text{if and only if} \quad \{p\}^1 \cap \mathrm{S}^{-1}_{\mathcal{A}^+} \subseteq \{q\}^1 \cap \mathrm{S}^{-1}_{\mathcal{A}^+},$$

that is,

$$\big\{a \in \mathrm{S}^{-1}_{\mathcal{A}^+} : \|p - a\| = 1\big\} \subseteq \big\{a \in \mathrm{S}^{-1}_{\mathcal{A}^+} : \|q - a\| = 1\big\}. \tag{4.2}$$

Proof. Assume that $q \leq p$. Consider $a \in \mathrm{S}^{-1}_{\mathcal{A}^+}$ satisfying $\|a - p\| = 1$. By parts (a) and (b) of Lemma 2.1, there exists $\omega \in \mathcal{PS}(\mathcal{A})$ such that $\omega(a) = 1$ and $\omega(p) = 0$. From this, we see that $\omega(q) = 0$ and hence $\omega(a - q) = 1$, which gives $\|a - q\| = 1$.

Conversely, assume that $q \not\leq p$. It follows from [14, Proposition 2.5] that there exist an AW^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ containing p and q, as well as Stonean spaces X and Y such that $\mathcal{B} \cong C(X) \oplus C(Y; \mathbb{M}_2)$. Let $p_1, q_1 \in \mathrm{P}_{C(X)}$ and $p_2, q_2 \in \mathrm{P}_{C(Y;\mathbb{M}_2)}$ be the projections with

$$p = p_1 + p_2 \quad \text{and} \quad q = q_1 + q_2.$$

Since $q \not\leq p$, either $q_1 \not\leq p_1$ or $q_2 \not\leq p_2$.

If $q_1 \not\leq p_1$, then $q_1 - q_1 p_1 \in \mathrm{P}_\mathcal{B} \setminus \{0\}$ and

$$a := \frac{1_\mathcal{B} + (q_1 - q_1 p_1)}{2} \in \mathrm{S}^{-1}_{\mathcal{B}^+}$$

satisfies $\|a - p\| = 1$, but $\|a - q\| \leq 1/2$, contradicting (4.2).

Now, assume that $q_2(y) \not\leq p_2(y)$ in $\mathbb{M}_2$ for some $y \in Y$. Let us first consider the situation that $q_2(y) = 1_{\mathbb{M}_2}$. Then $p_2(y)$ is either zero or a rank one projection. In any of these two cases, $\|1_{\mathbb{M}_2} - p_2(y)\| = 1$, and $\|1_{\mathbb{M}_2} - q_2(y)\| = 0$. The continuity of q_2 gives a clopen subset $U \subseteq Y$ containing y satisfying $\|1_{\mathbb{M}_2} - q_2(z)\| \leq 1/2$ whenever $z \in U$. If we set

$$a := \frac{1_{C(X)}}{2} + \mathbf{1}_U \otimes 1_{\mathbb{M}_2} + \mathbf{1}_{Y \setminus U} \otimes \frac{1_{\mathbb{M}_2}}{2},$$

then $a \in \mathrm{S}^{-1}_{\mathcal{B}^+}$, $\|a - p\| = 1$ (as $a(y) = 1_{\mathbb{M}_2}$), but $\|a - q\| \leq 1/2$, contradicting (4.2).

Next, we consider the situation when $q_2(y)$ is a rank one projection and $p_2(y) = 0$. Then one can find $v \in \mathrm{S}^{-1}_{\mathbb{M}_2^+}$ with $\|v - q_2(y)\| < 1/3$. The continuity of q_2 produces a clopen subset $V \subseteq Y$ containing y satisfying $\|v - q_2(z)\| \leq 1/3$ whenever $z \in V$. Then

$$a := \frac{1_{C(X)}}{2} + \mathbf{1}_V \otimes v + \mathbf{1}_{Y \setminus V} \otimes \frac{1_{\mathbb{M}_2}}{2} \in \mathrm{S}^{-1}_{\mathcal{B}^+},$$

satisfies $\|a - p\| = 1$ (as $a(y) = v$ and $p_2(y) = 0$) but $\|a - q\| \leq 1/2$, contradicting (4.2).

Finally, we consider the situation when $q_2(y)$ is a rank one projection and $p_2(y)$ is a rank one projection different from $q_2(y)$. We can find $w \in \mathrm{S}^{-1}_{\mathbb{M}_2^+}$ with

$$\|w - p_2(y)\| = 1 \quad \text{but} \quad \|w - q_2(y)\| < \gamma$$

for some $\gamma \in [0, 1)$. The same continuity argument as above gives

$$a := \frac{1_{C(X)}}{2} + \mathbf{1}_W \otimes w + \mathbf{1}_{Y \setminus W} \otimes \frac{1_{\mathbb{M}_2}}{2} \in \mathrm{S}^{-1}_{\mathcal{B}^+},$$

such that $\|a-p\| = 1$ (as $a(y) = w$) but $\|a-q\| \leq \max\{\gamma, 1/2\} < 1$, contradicting (4.2).

Consequently if $q \not\leq p$, then (4.2) fails. $\square$

Remark 4.3.

(a) By Lemmas 4.1 and 2.2, we know that the identity element $1 \in \mathrm{S}_{\mathcal{A}^+}$ is the unique element $a \in \mathrm{S}_{\mathcal{A}^+}$ satisfying the following properties:
- $\{a\} = (\{a\}^1)^1$;
- $(\mathrm{P}_{\mathcal{A}} \setminus \{0\}) \setminus \{a\} \subseteq \{a\}^1$ (i.e., $\|a-q\| = 1$ for every $q \in \mathrm{P}_{\mathcal{A}} \setminus \{a\}$);
- $\{b\}^1 \cup \{a\}^1 \neq \mathrm{S}_{\mathcal{A}^+}$, for every $b \in \mathrm{S}_{\mathcal{A}^+}$.

Observe that

$$E^1 = \left\{b \in \mathrm{S}_{\mathcal{A}^+} : (b, e) \in \mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}},\ \text{for every } e \in E\right\} \qquad (E \subseteq \mathrm{S}_{\mathcal{A}^+}),$$

and that, as a subset of $\mathrm{S}_{\mathcal{A}^+}$, the set $\mathrm{P}_{\mathcal{A}} \setminus \{0\}$ is also determined by $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$ (see Lemma 4.1). Consequently, one can locate the identity element 1 in $\mathrm{S}_{\mathcal{A}^+}$ via $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$.

(b) Part (a) above and the following equality (see Lemma 2.1(b))

$$\mathrm{S}^{-1}_{\mathcal{A}^+} = \{c \in \mathrm{S}_{\mathcal{A}^+} : \|1-c\| < 1\} = \mathrm{S}_{\mathcal{A}^+} \setminus \{1\}^1$$

tell us that one can also identify the subset $\mathrm{S}^{-1}_{\mathcal{A}^+}$ of $\mathrm{S}_{\mathcal{A}^+}$ through $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$.

(c) Part (b) above and Lemma 4.2 imply that one can determine the order relation on $\mathrm{P}_{\mathcal{A}}$ via $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$. This, together with Lemma 3.1, tells us that the orthogonality relation on $\mathrm{P}_{\mathcal{A}}$ is also determined by $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$.

Lemma 4.4. *Suppose that $\Lambda : \mathrm{S}_{\mathcal{A}^+} \to \mathrm{S}_{\mathcal{B}^+}$ is a bijection between the positive unit spheres of two AW*-algebras preserving pairs of points at diametrical distance, i.e., $(\Lambda \times \Lambda)(\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}) = \mathcal{D}_{\mathrm{S}_{\mathcal{B}^+}}$. Then the following statements hold:*

(a) $\Lambda(\mathrm{P}_{\mathcal{A}} \setminus \{0\}) = \mathrm{P}_{\mathcal{B}} \setminus \{0\}$

(b) $\Lambda(1) = 1$ *and* $\Lambda\big(\mathrm{S}^{-1}_{\mathcal{A}^+}\big) = \mathrm{S}^{-1}_{\mathcal{B}^+}$.

(c) *If we extend $\Lambda|_{\mathrm{P}_{\mathcal{A}} \setminus \{0\}}$ to a map $\bar{\Lambda} : \mathrm{P}_{\mathcal{A}} \to \mathrm{P}_{\mathcal{B}}$ with $\bar{\Lambda}(0) := 0$, then $\bar{\Lambda}$ is an order isomorphism.*

Proof.

(a) This follows from Lemma 4.1.

(b) This follows from Lemmas 4.1 and 2.2 (see Remark 4.3).

(c) Since $\Lambda(\mathrm{S}^{-1}_{\mathcal{A}^+}) = \mathrm{S}^{-1}_{\mathcal{B}^+}$, we know from Lemma 4.2 that $\Lambda(q) \leq \Lambda(p)$ if and only if $q \leq p$. $\square$

The following is a direct consequence of Lemma 4.4(c) as well as parts (b) and (d) of Proposition 3.2.

Proposition 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be AW*-algebras. Suppose that there is a bijection $\Lambda : \mathrm{S}_{\mathcal{A}^+} \to \mathrm{S}_{\mathcal{B}^+}$ preserving pairs of points at diametrical distance. Then the following statements hold:*

(a) *$\mathcal{A}$ is Jordan *-isomorphic to $\mathcal{B}$.*

(b) *If $\mathcal{A}$ has no type $\mathbf{I}_2$ summand, then $\Lambda|_{\mathrm{P}_{\mathcal{A}} \setminus \{0\}}$ extends to a Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

It is natural to ask whether one can improve Proposition 4.5(b) to the conclusion that Λ itself extends to a Jordan *-isomorphism. However, Example 4.6 tells us that this is not true in general.

Example 4.6. Let X be any non-empty set. We define a bijection $\Lambda : \mathrm{S}_{\ell^\infty(X)^+} \to \mathrm{S}_{\ell^\infty(X)^+}$ by

$$\Lambda(a)(x) := a(x)^3 \qquad (a \in \mathrm{S}_{\ell^\infty(X)^+}; x \in X).$$

Then Λ fixes every non-zero projection and preserves the diametral relations. Clearly, $\Lambda|_{\mathrm{P}_{\ell^\infty(X)}\setminus\{0\}}$ extends to the identity map from $\ell^\infty(X)$ onto $\ell^\infty(X)$. However, Λ itself does not extend to a Jordan *-isomorphism from $\ell^\infty(X)$ onto $\ell^\infty(X)$.

Lemma 4.7 below is an extension of [21, Lemma 2.3(c)] to AW^*-algebras (with basically the same proof). This, together with Remark 4.3(c) and Lemma 4.1, tells us that one can also determine whether a projection q is dominated by an invertible element $a \in \mathrm{S}_{\mathcal{A}^+}^{-1}$ through $\mathcal{D}_{\mathrm{S}_{\mathcal{A}^+}}$.

Lemma 4.7. *Let $\mathcal{A}$ be an AW^*-algebra. For $q \in \mathrm{P}_{\mathcal{A}} \setminus \{0,1\}$ and $a \in \mathrm{S}_{\mathcal{A}^+}^{-1}$, one has $q \leq a$ if and only if $\|1 - r - a\| = 1$ for every $r \in \mathrm{P}_{\mathcal{A}} \setminus \{0\}$ with $r \leq q$.*

Proof. The necessity follows from the fact that if $q \leq a$, then $0 \leq 1 - a \leq 1 - q \leq 1 - r$, and this implies $\|1 - r - a\| = \|r - (1 - a)\| = \max\{\|r\|, \|1 - a\|\} = 1$.

For the sufficiency, one needs to verify that $q(1-a) = 0$, when the said condition in the statement holds. Suppose on the contrary that $q(1 - a)q \neq 0$ (see Lemma 2.1(f)). There are $r \in \mathrm{P}_{\mathcal{A}} \setminus \{0,1\}$ and $\alpha \in (0,1]$ with

$$\alpha r \leq q(1 - a)q \leq q. \tag{4.3}$$

Hence, $r \leq q$. The said condition implies $\|r - (1 - a)\| = 1$. Since $a \in \mathcal{A}^{-1}$, parts (a) and (b) of Lemma 2.1 provides a pure state $\omega \in \mathcal{PS}(\mathcal{A})$ with

$$\omega(r) = 1 \quad \text{and} \quad \omega(1 - a) = 0. \tag{4.4}$$

The first equality above and Lemma 2.1(c) give (note that $\alpha r \leq r(1-a)r$, because of (4.3))

$$\omega(1 - a) = \omega(r(1 - a)r) \geq \alpha\omega(r) = \alpha > 0,$$

which contradicts the second equality in Relation (4.4). $\square$

Note that the invertibility of a in the above lemma is essential, because the backward implication fails when $\mathcal{A} = \mathbb{C}^3$, $q = (1, 0, 0)$ and $a = (0, 0, 1)$.

Theorem 4.8 below extends [37, Theorem 3.6] and [21, Theorem 4.5] to general AW^*-algebras (while in [37, Theorem 3.6] and [21, Theorem 4.5], the corresponding results are proved for atomic von Neumann algebras and for type **I** finite von Neumann algebras with bounded dimensions of irreducible representations, respectively). Some arguments in the proof are inspired by both [21] and [37].

Theorem 4.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be AW^*-algebras. If $\Lambda : \mathrm{S}_{\mathcal{A}^+} \to \mathrm{S}_{\mathcal{B}^+}$ is a surjective isometry, then Λ extends to a Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. By Lemma 4.4(c) and Theorem 3.4, the bijection $\Lambda|_{\mathrm{P}_{\mathcal{A}}\setminus\{0\}} : \mathrm{P}_{\mathcal{A}} \setminus \{0\} \to \mathrm{P}_{\mathcal{B}} \setminus \{0\}$ extends to a Jordan *-isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$. It remains to show that $\Psi|_{\mathrm{S}_{\mathcal{A}^+}} = \Lambda$.

In fact, we know from Lemma 4.4(b) that $\Lambda(1) = 1$. Moreover, by Lemmas 4.2, 4.4(b) and 4.7,

$$p \leq a \quad \text{if and only if} \quad \Lambda(p) \leq \Lambda(a) \quad (p \in \mathrm{P}_A \setminus \{0,1\}; a \in \mathrm{S}_{\mathcal{A}^+}^{-1}), \tag{4.5}$$

because $\Lambda(1-r) = \Psi(1-r) = 1 - \Lambda(r)$ and $\Lambda(r) = \Psi(r) \leq \Lambda(p)$ when $r \in \mathrm{P}_A \setminus \{0\}$ with $r \leq p$.

Consider an arbitrary projection $p_0 \in \mathrm{P}_{\mathcal{A}} \setminus \{0,1\}$, and set $q_0 := \Lambda(p_0)$. Write

$$\mathcal{A}[p_0] := (1-p_0)\mathcal{A}(1-p_0) \quad \text{and} \quad \mathcal{B}[q_0] := (1-q_0)\mathcal{B}(1-q_0).$$

We claim that there is a Jordan *-isomorphism $\Phi^{p_0} : \mathcal{A}[p_0] \to \mathcal{B}[q_0]$ satisfying

$$\Lambda(p_0 + a) = q_0 + \Phi^{p_0}(a) \quad \big(a \in \mathrm{B}_{\mathcal{A}[p_0]^+}\big). \tag{4.6}$$

Indeed, it follows from Lemma 4.4(b), Relation (4.5) and Lemma 2.1(e) that

$$\Lambda\big(p_0 + \mathrm{B}^{-1}_{\mathcal{A}[p_0]^+}\big) = q_0 + \mathrm{B}^{-1}_{\mathcal{B}[q_0]^+}.$$

Since Λ is isometric and $\mathrm{B}^{-1}_{\mathcal{A}[p_0]^+}$ is norm dense in $\mathrm{B}_{\mathcal{A}[p_0]^+}$, we have

$$\Lambda\big(p_0 + \mathrm{B}_{\mathcal{A}[p_0]^+}\big) = q_0 + \mathrm{B}_{\mathcal{B}[q_0]^+}.$$

This gives a surjective isometry $\Phi^{p_0} : \mathrm{B}_{\mathcal{A}[p_0]^+} \to \mathrm{B}_{\mathcal{B}[q_0]^+}$ satisfying Relation (4.6). Using [22, Theorem 5] (notice that B_{D^+} is the closure of its interior in D_{sa} for any unital C^*-algebra D), we know that Φ^{p_0} extends to a bijective affine isometry, again denoted by Φ^{p_0}, from $\mathcal{A}[p_0]_{\mathrm{sa}}$ onto $\mathcal{B}[q_0]_{\mathrm{sa}}$. Since $\Phi^{p_0}(0) = 0$, one sees that $\Phi^{p_0} : \mathcal{A}[p_0]_{\mathrm{sa}} \to \mathcal{B}[q_0]_{\mathrm{sa}}$ is a real linear surjective isometry. Hence, Φ^{p_0} extends to a Jordan *-isomorphism $\Phi^{p_0} : \mathcal{A}[p_0] \to \mathcal{B}[q_0]$ (see [17]).

Next, we assert that Φ^{p_0} satisfies

$$\Phi^{p_0}(e) = \Lambda(e) = \Psi(e) \quad (e \in \mathrm{P}_{\mathcal{A}[p_0]} \setminus \{0\}). \tag{4.7}$$

In fact, consider $e \in \mathrm{P}_{\mathcal{A}[p_0]} \setminus \{0\}$. The equality $\Lambda(e) = \Psi(e)$ comes from the definition of Ψ. Moreover, it follows from (4.6) that

$$\Lambda(p_0 + e) = q_0 + \Phi^{p_0}(e).$$

On the other hand, as $p_0 + e \in \mathrm{P}_{\mathcal{A}} \setminus \{0\}$, we have

$$\Lambda(p_0 + e) = \Psi(p_0 + e) = \Psi(p_0) + \Psi(e) = q_0 + \Lambda(e).$$

We thus conclude that $\Phi^{p_0}(e) = \Lambda(e)$, as asserted.

Finally, consider $a \in \mathrm{S}_{\mathcal{A}^+}$ and $\epsilon > 0$. One can find $\alpha_1, \ldots, \alpha_n \in (0,1)$ as well as pairwise orthogonal projections $p_0, p_1, \ldots, p_n \in \mathrm{P}_{\mathcal{A}} \setminus \{0\}$ such that $\|a - (p_0 + \sum_{i=1}^n \alpha_i p_i)\| < \epsilon$. As both Λ and Ψ are isometric, we have

$$\Big\|\Lambda(a) - \Lambda\big(p_0 + \sum\nolimits_{i=1}^n \alpha_i p_i\big)\Big\| < \epsilon \quad \text{and} \quad \Big\|\Psi(a) - \Psi\big(p_0 + \sum\nolimits_{i=1}^n \alpha_i p_i\big)\Big\| < \epsilon.$$

It is clear that $p_1, \ldots, p_n \in \mathrm{P}_{\mathcal{A}[p_0]} \setminus \{0\}$ and $\sum_{i=1}^n \alpha_i p_i \in \mathrm{B}_{\mathcal{A}[p_0]^+}$. Thus, Relations (4.6) and (4.7) as well as the definition of Ψ imply that

$$\begin{aligned}\Lambda\big(p_0 + \sum\nolimits_{i=1}^n \alpha_i p_i\big) &= q_0 + \Phi^{p_0}\big(\sum\nolimits_{i=1}^n \alpha_i p_i\big) = \Lambda(p_0) + \sum\nolimits_{i=1}^n \alpha_i \Psi(p_i) \\ &= \Psi(p_0) + \sum\nolimits_{i=1}^n \alpha_i \Psi(p_i) = \Psi\big(p_0 + \sum\nolimits_{i=1}^n \alpha_i p_i\big).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get $\Lambda(a) = \Psi(a)$, as required. □

By Lemma 3.5, when both $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, Proposition 4.5 and Theorem 4.8 can be restated as

Corollary 4.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras with projection lattices $\mathrm{P}_{\mathcal{A}}, \mathrm{P}_{\mathcal{B}}$ and positive unit spheres $\mathrm{S}_{\mathcal{A}^+}, \mathrm{S}_{\mathcal{B}^+}$, respectively. Let $\Lambda : \mathrm{S}_{\mathcal{A}^+} \to \mathrm{S}_{\mathcal{B}^+}$ be a bijection preserving pairs of points at diametrical distance.*

(a) *$\mathcal{A}$ is real linear *-isomorphic to $\mathcal{B}$.*

(b) *If $\mathcal{A}$ has no type $\mathbf{I}_2$ summand, then $\Lambda|_{\mathrm{P}_{\mathcal{A}}\setminus\{0\}}$ extends to a real linear *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

(c) *If $\Lambda : \mathrm{S}_{\mathcal{A}^+} \to \mathrm{S}_{\mathcal{B}^+}$ is a surjective isometry, then Λ extends to a real linear *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

While Theorem 4.8 indicates that Problem 1.2 is likely to have an affirmative answer when E and F are self-adjoint parts of C^*-algebras, we end this paper with Problem 4.10. Proposition 4.5(a) tells us that this problem has an affirmative answer for AW^*-algebras.

Problem 4.10. Let A and B be C^*-algebras. Suppose that there exists a bijection from S_{A^+} onto S_{B^+} preserving pairs of points at diametrical distance. Is A Jordan *-isomorphic to B?

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