THIN BELLS IN L^p-SPACES AS JORDAN INVARIANTS FOR VON NEUMANN ALGEBRAS

CHI-WAI LEUNG, CHI-KEUNG NG, AND NGAI-CHING WONG

Abstract. Extending the main result in [10], we show that for any fixed $p \in [1,\infty]$ and any $\epsilon \in (0,1]$, the metric space

$$\{S^{\frac{1}{p}} \in L^p_+(M) : 1-\epsilon \leq \|S^{\frac{1}{p}}\| \leq 1\}$$

is a complete Jordan *-invariant for a von Neumann algebra M. Furthermore, in the case when $p \in (1,\infty)$, if $M \ncong \mathbb{C}$ and is a semifinite algebra with no type I_2 summand (or is a hyperfinite algebra with no type I_2 summand), then for any von Neumann algebra N and any metric preserving bijection

$$\Phi : \{S \in L^p_+(M) : 1-\epsilon \leq \|S^{\frac{1}{p}}\| \leq 1\} \to \{T \in L^p_+(N) : 1-\epsilon \leq \|T^{\frac{1}{p}}\| \leq 1\},$$

there is a Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Phi(S^{\frac{1}{p}}) = \Theta_*(S)^{\frac{1}{p}}$.

1. Introduction and Notation

It is well-known that several partial structures of a von Neumann algebra can serve as complete Jordan *-invariants of a von Neumann algebra (see e.g. [7, Theorem 2], [7, Corollary 5], [8, Theorem 4.5], [18, Theorem 3] and [5, Théorème 3.3]). In particular, generalizing results in [14], [20] and [21], D. Sherman showed in [15] that the metric space structure of the non-commutative L^p-space is a complete Jordan *-invariant for the underlying von Neumann algebra, when $p \in [1,\infty] \setminus \{2\}$ (observe that the non-commuative L^2-space of any infinite dimensional von Neumann algebra with separable predual is ℓ^2).

Since any bijective isometry between normed spaces is automatically affine, it is natural to ask whether it is possible to obtain a "smaller invariant" by excluding those part that could be recover from a smaller subset of the non-commuative L^p-space. Alone this line, we show in [10] that, for each $p \in [1,\infty]$, the positive contractive part of the non-commuative L^p-space, again as a metric space, is a complete Jordan *-invariant for the underlying von Neumann algebra (note the different here that one can include the case of $p = 2$, since the cone of the L^2-space encodes some information that cannot be recovered from the normed space structure).

Continuing with this philosophy, we will show in Section 2 of this article the following result concerning an arbitrarily thin bell $L^p_+(M)^{\beta+\epsilon}_{\beta-\epsilon} := \{R \in L^p_+(M) : \beta-\epsilon \leq \|R\| \leq \beta+\epsilon\}$ as a complete Jordan invariant.

Theorem 1.1. *Let $p \in [1,\infty]$ and $\beta \in \mathbb{R}_+ \setminus \{0\}$ and $\epsilon \in (0,\beta]$. If there is a metric preserving bijection $\Phi : L^p_+(M)^{\beta+\epsilon}_{\beta-\epsilon} \to L^p_+(N)^{\beta+\epsilon}_{\beta-\epsilon}$, then M and N are Jordan *-isomorphic.*

In the case of $p = 1$, this is proved by showing that some elements with norm β is mapped to elements with norm β in an "orthogonality support preserving way", we then use a result of Dye to obtain the conclusion. In the case of $p = \infty$, we show that some points in the interior of the bell is mapped to

Date: May 8, 2017.

2010 *Mathematics Subject Classification.* Primary: 46L10, 46L52; Secondary: 54E35.

Key words and phrases. non-commutative L^p-spaces; positive contractive elements; metric spaces; bijective isometries; Jordan *-isomorphisms.

the interior of the other bell, and then use a "stronger form of the Mazur-Ulam theorem" and a result of Kadison to get the Jordan *-isomorphism. In the case of $p \in (1,\infty)$, we use the strict convexity to verify that the map Φ is "partially homogeneous" and the canonical extension to the whole cone is also isometric. Then we use some equality related to the non-commutative Clarkson inequality to a "biorthogonality preserving map" between the normal state spaces, and employ a result in [9] to finish the proof.

The proof of the case $p \in (1,\infty)$ can be generalized to a statement concerning extension of maps between the bells to that of between the cones. From this, we have the following.

> Let $p \in (1,\infty) \setminus \{2\}$. If $\epsilon \in (0,1]$ and
> $$\Phi : \{S \in L^p(M) : 1-\epsilon \leq \|S\| \leq 1\} \to \{T \in L^p(N) : 1-\epsilon \leq \|T\| \leq 1\}$$
> is a metric preserving bijection, then one can find a Jordan *-isomorphism $\Theta : N \to M$ with Φ is defined by Θ in a canonical way.

On the other hand, it was asked in [10] whether a metric preserving bijection from the positive contractive part of the non-commuative L^p-space of one von Neumann algebra to that of another von Neumann algebra is defined by a Jordan *-isomorphism in a canonical way. Although the above quoted statement is true, there seems to have no way to obtain this strong form from this statement in the case when $p \in (1,\infty) \setminus \{2\}$. Nevertheless, we give in, Section 3, an affirmative answer to this question in the case of $p \in (1,\infty)$ when the algebra satisfying a condition called EP_1 (which is true when the algebra is semifinite algebras and has no type I_2 summand). In fact, we give a more general result as follows:

Theorem 1.2. *Let $p \in (1,\infty)$ and $\beta \in \mathbb{R}_+ \setminus \{0\}$ and $\epsilon \in (0,\beta]$. Let M and N be von Neumann algebras such that M has EP_1 and $M \not\cong \mathbb{C}$. Suppose that $\Phi : L^p_+(M)^{\beta+\epsilon}_{\beta-\epsilon} \to L^p_+(N)^{\beta+\epsilon}_{\beta-\epsilon}$ is a metric preserving surjection. There is a Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Phi(R^{\frac{1}{p}}) = \Theta_*(R)^{\frac{1}{p}}$ ($R^{\frac{1}{p}} \in L^p_+(M)^{\beta}_{\alpha}$).*

Let us set some notations and recall some facts in the remainder of this section. Throughout this article, M and N are von Neumann algebras with predual M_* and N_*, respectively. We use $\mathcal{P}(M)$ to denote the set of projections in M. We fix a normal semifinite faithful weight φ on M and consider the modular automorphism group α corresponding to φ. Since the von Neumann algebra crossed product $\check{M} := M \bar{\rtimes}_\alpha \mathbb{R}$ is semi-finite, we choose a normal faithful semi-finite trace τ on $\check{M}$. Denote by $L^0(\check{M},\tau)$ the completion M under the vector topology defined by a neighborhood basis at 0 of the form

$$U(\epsilon,\delta) := \{x \in \check{M} : \|xp\| \leq \epsilon \text{ and } \tau(1-p) \leq \delta, \text{ for a projection } p \in \check{M}\}.$$

The *-algebra structure on $\check{M}$ extends to a *-algebra structure on $L^0(\check{M},\tau)$.

If M is a von Neumann algebra on a Hilbert space $\mathfrak{H}$, then elements in $L^0(\check{M},\tau)$ can be regarded as closed operators on $L^2(\mathbb{R};\mathfrak{H})$. More precisely, let T be a densely defined closed operator on $L^2(\mathbb{R};\mathfrak{H})$ affiliated with $\check{M}$ and $|T|$ be its absolute value with spectral measure $E_{|T|}$. Then T corresponds uniquely to an element in $L^0(\check{M},\tau)$ if and only if $\tau\big(1-E_{|T|}([0,\lambda])\big) < \infty$ when λ is large. Conversely, every element in $L^0(\check{M},\tau)$ comes a closed operator in this way. Under this identification, the *-operation on $L^0(\check{M},\tau)$ coincides with the adjoint. The addition and the multiplication on $L^0(\check{M},\tau)$ are the closures of the corresponding operations for closed operators. We denote by $L^0_+(\check{M},\tau)$ the set of all positive self-adjoint operators in $L^0(\check{M},\tau)$.

The dual action $\hat{\alpha} : \mathbb{R} \to \mathrm{Aut}(\check{M})$ extends to an action on $L^0(\check{M},\tau)$. For any $p \in [1,\infty]$, we set

$$L^p(M) := \{T \in L^0(\check{M},\tau) : \hat{\alpha}_s(T) = e^{-s/p}T, \text{ for all } s \in \mathbb{R}\}$$

(where $e^{-s/\infty}$ means 1). Then $L^\infty(M)$ coincides with the subalgebra M of $\check{M} \subseteq L^0(\check{M}, \tau)$. Moreover, if $T \in L^0(\check{M}, \tau)$ and $T = u|T|$ is the polar decomposition, then $T \in L^p(M)$ if and only if $|T| \in L^p(M)$. Denote by $L^p_{\rm sa}(M)$ the set of all self-adjoint operators in $L^p(M)$ and put $L^p_+(M) := L^p(M) \cap L^0_+(\check{M}, \tau)$.

When $q \in (0, \infty)$, the Mazur map

$$S \mapsto S^{1/q} \qquad (S \in L^0_+(\check{M}, \tau))$$

restricts to a bijection from $L^1_+(M)$ onto $L^q_+(M)$. Since we use this connection between $L^1_+(M)$ onto $L^q_+(M)$ a lot, *elements in $L^q_+(M)$ will always be written in the form $S^{1/q}$ (for a unique $S \in L^1_+(M)$).*

As in the literature,

we identify $\big(L^1(M), L^1_+(M)\big)$ with (M_*, M_*^+) as ordered vector spaces throughout this article.

Hence, $\big(L^1(M), L^1_+(M)\big)$ is an ordered Banach space with norm $\|\cdot\|_1$. When $p \in (1, \infty)$, the function:

$$\|T\|_p := \big\||T|^p\big\|_1^{1/p}$$

is a norm on $L^p(M)$, so that $\big(L^p(M), L^p_+(M)\big)$ becomes an ordered Banach space. It is well-known that this ordered Banach space is independent of the choices of φ and τ.

For $T \in L^1_+(M)$, we denote by $\mathbf{s}_T \in \mathcal{P}(M)$ the "support of T". Recall that a map Λ from a subset E of $L^1_+(M)$ to $L^1_+(N)$ is said to be *orthogonality preserving* if for $R, T \in E$, one has

$$\mathbf{s}_R \cdot \mathbf{s}_T = 0 \quad \text{implies} \quad \mathbf{s}_{\Lambda(R)} \cdot \mathbf{s}_{\Lambda(T)} = 0. \tag{1.1}$$

Let us recall the following result. The first statement of part (a) is a reformulation of [12, Proposition A.6] and the second statement follows from [12, Fact 1.3], while part (b) is very well-known.

Lemma 1.3. *Let $R, T \in L^1_+(M)$.*

(a) Suppose that $p \in (1, \infty)$. Then $\mathbf{s}_R \cdot \mathbf{s}_T = 0$ if and only if $\|R^{\frac{1}{p}} + T^{\frac{1}{p}}\|_p^p = \|R^{\frac{1}{p}}\|_p^p + \|T^{\frac{1}{p}}\|_p^p$. In this case, one also has $\|R^{\frac{1}{p}} - T^{\frac{1}{p}}\|_p^p = \|R^{\frac{1}{p}}\|_p^p + \|T^{\frac{1}{p}}\|_p^p$.

(b) $\mathbf{s}_R \cdot \mathbf{s}_T = 0$ if and only if $\|R - T\|_1 = \|R\|_1 + \|T\|_1$.

From this, one sees that if a map $\Lambda : L^1_+(M) \to L^1_+(N)$ satisfies $\|\Lambda(R)\| = \|R\|$ and $\Lambda(R+T) = \Lambda(R) + \Lambda(T)$ for any $R, T \in L^1_+(M)$ with $\mathbf{s}_R \cdot \mathbf{s}_T = 0$, then Λ is orthogonality preserving.

Our second lemma is well-known, but since we cannot find the exact reference in the literature, we give their justification here.

Lemma 1.4. *(a) $S \mapsto S^{1/p}$ is a homeomorphism from $L^1_+(M)$ onto $L^p_+(M)$, for any $p \in (1, \infty)$.*

(b) Let $q \in (0, \infty)$. If $R, T \in L^1(M)_+$ with $\mathbf{s}_R \mathbf{s}_T = 0$, then $(R+T)^q = R^q + T^q$.

Proof. (a) It follows from [13, Lemma 2.1] that

$$\|R^{1/p} - T^{1/p}\|_p^p \leq \|R - T\|_1 \qquad (R, T \in L^1(M)_+).$$

On the other hand, it follows from [13, Corollary 2.3] that

$$\|R - T\|_1 \leq 3p\|R^{1/p} - T^{1/p}\|_p \max\big\{\|R^{1/p}\|_p, \|T^{1/p}\|_p\big\}^{p-1} \qquad (R, T \in L^1_+(M)).$$

These give the required statement.

(b) Let $\mathfrak{K}_R := \mathbf{s}_R(L^2(\mathbb{R}; \mathfrak{H}))$ and $\mathfrak{K}_T := \mathbf{s}_T(L^2(\mathbb{R}; \mathfrak{H}))$. Let $\mathfrak{K}_0$ be the orthogonal complement of $\mathfrak{K}_R + \mathfrak{K}_T$. As $R = \mathbf{s}_R R \mathbf{s}_R$, the restriction, R_1, of R on $\mathfrak{K}_R$ is a densely defined positive self-adjoint operator. The same is true for the restriction, T_1, of T on $\mathfrak{K}_T$. One may then identify R, T and $R+T$ with

$R_1 \oplus 0_{\mathfrak{K}_T} \oplus 0_{\mathfrak{K}_0}$, $0_{\mathfrak{K}_R} \oplus T_1 \oplus 0_{\mathfrak{K}_0}$ and $R_1 \oplus T_1 \oplus 0_{\mathfrak{K}_0}$, respectively. Thus, $R^q + T^q$ can be identified with the closed operator $R_1^q \oplus T_1^q \oplus 0_{\mathfrak{K}_0}$, which clearly coincides with $(R+T)^q$. □

2. Positive bells as a complete Jordan invariant

If X is a normed space and $E \subseteq X$ is a subset, we set

$$E_\alpha^\beta := \{x \in E : \alpha \leq \|x\| \leq \beta\} \quad \text{for any } \alpha \leq \beta \neq 0 \text{ in } \mathbb{R}_+.$$

For simplicity, we may use $\|\cdot\|$ instead of $\|\cdot\|_p$ to denote the norm on $L^p(M)$, if no confusion arises.

We say that a projection $r \in \mathcal{P}(M)$ is *σ-finite* if there exists $R \in L^1_+(M)$ such that $r = \mathbf{s}_R$. The set of all σ-finite projections in M will be denoted by $\mathcal{P}_0(M)$. It is well-known that for any projection $p \in \mathcal{P}(M)$ is the supremum in $\mathcal{P}(M)$ of the collection $\{r \in \mathcal{P}_0(M) : r \leq p\}$.

Proposition 2.1. *Let $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$. If there is a metric preserving bijection $\Phi : L^1_+(M)_\alpha^\beta \to L^1_+(N)_\alpha^\beta$, then M and N are Jordan *-isomorphic.*

Proof. Let $L^1_\beta(M) := \{R \in L^1_+(M)_\beta^\beta : \mathbf{s}_R \neq 1\}$. For any $R \in L^1_+(M)_\alpha^\beta$, it is easy to see, using Lemma 1.3(b), that $R \in L^1_\beta(M)$ if and only if there exists $T \in L^1_+(M)_\alpha^\beta$ such that $\|R - T\| = 2\beta$. In this case, $T \in L^1_\beta(M)$ and $\mathbf{s}_R \cdot \mathbf{s}_T = 0$. Hence, by considering Φ and Φ^{-1}, one has $\Phi(L^1_\beta(M)) = L^1_\beta(N)$.

Let us formally define a map

$$\Delta : \mathcal{P}_0(M) \setminus \{1\} \to \mathcal{P}_0(N) \setminus \{1\}$$

by $\Delta(p) := \mathbf{s}_{\Phi(R)}$, where $R \in L^1_\beta(M)$ satisfying $\mathbf{s}_R = p$. To show that Δ is well-defined, let us first consider another element $R' \in L^1_\beta(M)$ with $\mathbf{s}_{R'} = p$. Pick any projection $q \in \mathcal{P}_0(N)$ and any operator $T \in L^1_\beta(M)$ such that $\mathbf{s}_{\Phi(R)} \cdot q = 0$ and $\mathbf{s}_{\Phi(T)} = q$. Since

$$\|R - T\| = \|\Phi(R) - \Phi(T)\| = 2\beta,$$

we know from Lemma 1.3(b) that $p \cdot \mathbf{s}_T = 0$ and hence we have $\|\Phi(R') - \Phi(T)\| = \|R' - T\| = 2\beta$, which gives $\mathbf{s}_{\Phi(R')} \cdot q = 0$. From this, we conclude that $\mathbf{s}_{\Phi(R')} = \mathbf{s}_{\Phi(R)}$, and Δ is well-defined. Suppose that $p_1, p_2 \in \mathcal{P}_0(M) \setminus \{1\}$ such that $p_1 \cdot p_2 = 0$. If $R_1, R_2 \in L^1_\beta(M)$ satisfying $\mathbf{s}_{R_i} = p_i$ for $i = 1, 2$, then $\|\Phi(R_1) - \Phi(R_2)\| = 2\beta$, which gives $\Delta(p_1) \cdot \Delta(p_2) = 0$.

Now, we extend Δ to $\bar{\Delta} : \mathcal{P}(M) \to \mathcal{P}(N)$ by setting $\bar{\Delta}(1) = 1$ and $\bar{\Delta}(p)$ to be the supremum in $\mathcal{P}(N)$ of the $\{\Delta(p') : p' \in \mathcal{P}_0(N); p' \leq p\}$. Employing the argument as in [9], one can show that $\bar{\Delta}$ is an orthoisomorphism in the sense of Dye (see [6]), and the conclusion follows from a corollary of the main result of [6] (more precisely, see [9, Proposition 2.2]). □

Proposition 2.2. *Let $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$. If there is a metric preserving bijection $\Phi : L^\infty_+(M)_\alpha^\beta \to L^\infty_+(N)_\alpha^\beta$, then M and N are Jordan *-isomorphic.*

Proof. As in the Section 1, we identify $L^\infty_+(M)_\alpha^\beta$ and $L^\infty_+(N)_\alpha^\beta$ with $(M_+)_\alpha^\beta$ and $(N_+)_\alpha^\beta$ respectively. For any $y \in N_{\rm sa}$ and $r > 0$, we consider $D_N(y, r)$ to be the open ball with centre y and radius r. If in case $y \in (N_+)_\alpha^\beta$, we set

$$D_N^{\alpha,\beta}(y, r) := D_N(y, r) \cap (N_+)_\alpha^\beta.$$

For any $x \in (N_+)_0^\beta$, by considering the unital C^*-subalgebra of N generated by x, one can see easily that x belongs to the closed ball B with centre $\beta/2 \in N_+$ and radius $\beta/2$. Conversely, by considering unital C^*-subalgebras of N generated by single elements in B, one sees that $(N_+)_0^\beta = B$. This shows that $D_N(\beta/2, \beta/2)$ is dense in $(N_+)_0^\beta$. Let us put

$$\mathcal{O} := D_N(\beta/2, \beta/2) \setminus (N_+)_0^\alpha, \quad \mathcal{B}_1 := \{y \in N_{\rm sa} : \|y - \beta/2\| = \beta/2; \|y\| > \alpha\} \quad \text{and} \quad \mathcal{B}_2 := (N_+)_\alpha^\alpha.$$

Clearly, $\mathcal{O}$ is open in $N_{\rm sa}$ and $(N_+)^\beta_\alpha = \mathcal{O} \cup \mathcal{B}_1 \cup \mathcal{B}_2$.

Consider $b \in (N_+)^\beta_\alpha \setminus \mathcal{O}$ and $r > 0$. If $b \in \mathcal{B}_1$ and r is small enough, then

$$D_N^{\alpha,\beta}(b,r) = D_N(b,r) \cap (N_+)^\beta_0$$

and we know from the density of $D_N(\beta/2,\beta/2)$ in $(N_+)^\beta_0$ that $D_N^{\alpha,\beta}(b,r) \cap \mathcal{O} \neq \emptyset$. Suppose that $b \in \mathcal{B}_2$ and $r < \beta - \alpha$. Then $(1 + r/2\alpha)b \in (N_+)^\beta_\alpha$. If $(1 + r/2\alpha)b \notin \mathcal{O}$, then $(1 + r/2\alpha)b \in \mathcal{B}_1$ and the above tells us that $D_N^{\alpha,\beta}\big((1 + r/2\alpha)b, r'\big) \cap \mathcal{O} \neq \emptyset$ when r' is small enough, and hence $D_N^{\alpha,\beta}(b,r) \cap \mathcal{O} \neq \emptyset$. The above shows that $\mathcal{O}$ is dense in $(N_+)^\beta_\alpha$.

Now, we want to show that $c \in (M_+)^\beta_\alpha$ and $t > 0$ such that $D_M(c,t) \subseteq (M_+)^\beta_\alpha$ and $\Phi\big(D_M(c,t)\big)$ is an open subset of $N_{\rm sa}$. Indeed, suppose that a is an element in the interior of $(M_+)^\beta_\alpha$ and $s > 0$. If $\Phi(a) \in \mathcal{O}$, then we can take $c = a$ and $t = s$. If $\Phi(a) \notin \mathcal{O}$, then by the density of $\mathcal{O}$ in $(N_+)^\beta_\alpha$, there exist $b \in \mathcal{O} \cap D_N^{\alpha,\beta}(\Phi(a), s)$. There is $t > 0$ with

$$D_N(b,t) \subseteq D_N^{\alpha,\beta}(\Phi(a), s).$$

Then $D_M\big(\Phi^{-1}(b), t\big) \subseteq (M_+)^\beta_\alpha$ and $\Phi\big(D_M(\Phi^{-1}(b), t)\big) = D_N(b,t)$. Consequently, [3, Theorem 14.1] tells us that $\Phi|_{D_M(c,t)}$ extends to bijective isometry from $M_{\rm sa}$ onto $N_{\rm sa}$, and [7, Theorem 2] gives the required conclusion. □

For the case of $p \in (1,\infty)$, we need two lemmas. The following lemma is probably known. In fact, it was first proved by Baker in [2] that any metric preserving map from a normed space to a strictly convex normed space is automatically affine. Our generalization here use a different proof than the one in [2], which seemingly cannot be extended to obtain our lemma.

Lemma 2.3. *Let X and Y be two real normed spaces with Y being strictly convex. Suppose that E is a (not necessarily convex) subset of X and $f : E \to Y$ is a metric preserving map. Then for any $x, y \in E$, one has*

$$f(sx + (1-s)y) = sf(x) + (1-s)f(y) \quad \text{whenever } s \in (0,1) \text{ satisfying } sx + (1-s)y \in E. \tag{2.1}$$

Proof. Notice that

$$\begin{aligned}\big\|\big(f(x) - f(y)\big) - \big(f(sx + (1-s)y) - f(y)\big)\big\| &= \|x - (sx + (1-s)y)\| \ = \ (1-s)\cdot\|x - y\| \\ &= \|f(x) - f(y)\| - \|f(sx + (1-s)y) - f(y)\| \end{aligned}\tag{2.2}$$

Hence, the strict convexity of Y produces $\delta \in \mathbb{R}_+$ such that

$$\big(f(x) - f(y)\big) - \big(f(sx + (1-s)y) - f(y)\big) = \delta\big(f(sx + (1-s)y) - f(y)\big).$$

It now follows again from (2.2) that

$$(1-s)\cdot\|x-y\| = \big\|\big(f(x) - f(y)\big) - \big(f(sx + (1-s)y) - f(y)\big)\big\| \ = \ \delta s \cdot \|x - y\|,$$

and so $\delta = (1-s)/s$. Hence, $f(sx + (1-s)y) = sf(x) + (1-s)f(y)$ as required. □

Note that if E is a subset of the unit sphere of a strictly convex normed space X, then any map from E to any normed space Y will satisfy (2.1).

Our second lemma is also easy, but again, we present its full argument here.

Lemma 2.4. *Let X and Y be two normed spaces, and let $K \subseteq X$ and $L \subseteq Y$ be proper cones. If $\beta \in \mathbb{R}_+ \setminus \{0\}$ and $f : K_0^\beta \to L_0^\beta$ is an affine map (not necessarily surjective) with $f(0) = 0$, then f extends uniquely to an affine map $\bar f$ from K onto L. If, in addition, f preserves metric, then so is $\bar f$.*

Proof. For each $m \in \mathbb{N}$, we set $K^m := K_0^{m\beta}$ as well as $L^m := L_0^{m\beta}$, and we define $f^m : K^m \to L^m$ by

$$f^m(z) := mf(z/m) \qquad (z \in K^m).$$

As f is affine and $f(0) = 0$, we know that f^m is affine and that $f^{m+1}|_{K^m} = f^m$, for any $m \in \mathbb{N}$. This produces an affine map $\bar{f} : K \to L$ such that $\bar{f}(z) = f^m(z)$ whenever $z \in K^m$ for some $m \in \mathbb{N}$. Clearly, there exist at more one affine map extending f. Furthermore, if we assume that f is metric preserving, then so is f^m and hence $\bar{f}$ preserves metric. □

Now, we have the following extension of [10, Theorem 3.1], in the case when $p \in (1, \infty)$. Let us first recall the well-known fact that $L^p_{\mathrm{sa}}(M)$ is strictly convex (see e.g., Section 5 of [11]).

Proposition 2.5. *Let $p \in (1, \infty)$ and $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$. If there is a metric preserving bijection $\Phi : L^p_+(M)^\beta_\alpha \to L^p_+(N)^\beta_\alpha$, then M and N are Jordan *-isomorphic.*

Proof. If $M \cong \mathbb{C}$, then $L^p_+(M)^\beta_\alpha$ is a closed and bounded interval. As Φ is a metric preserving bijection, $L^p_+(N)^\beta_\alpha$ is also a closed and bounded interval, which implies that $N \cong \mathbb{C}$. The corresponding conclusion holds when $N \cong \mathbb{C}$. Therefore, we only consider the cases when $M \ncong \mathbb{C}$ and $N \ncong \mathbb{C}$.

Let us first show that

$$\Phi(L^p_+(M)^\beta_\beta) = L^p_+(N)^\beta_\beta \quad \text{and} \quad \Phi(L^p_+(M)^\alpha_\alpha) = L^p_+(N)^\alpha_\alpha. \tag{2.3}$$

In fact, consider an arbitrary element $S^{\frac{1}{p}} \in L^p_+(M)^\beta_\beta$. If $\|\Phi(S^{\frac{1}{p}})\| \in (\alpha, \beta)$, then $\Phi(S^{\frac{1}{p}})$ is the mid-point of two distinct elements in $L^p_+(N)^\beta_\alpha$ and by Lemma 2.3 (when applying to Φ^{-1}), the element $S^{\frac{1}{p}} \in L^p_+(M)^\beta_\beta$ is also the mid-point of two distinct elements in $L^p_+(M)^\beta_\alpha$, which is impossible (as $L^p_{\mathrm{sa}}(M)$ is strictly convex). Consequently, $\Phi(L^p_+(M)^\beta_\beta) \subseteq L^p_+(N)^\alpha_\alpha \cup L^p_+(N)^\beta_\beta$. Moreover, since $L^p_+(M)^\beta_\beta$ is path-connected and Φ is continuous, one sees that

$$\text{either} \quad \Phi(L^p_+(M)^\beta_\beta) \subseteq L^p_+(N)^\alpha_\alpha \quad \text{or} \quad \Phi(L^p_+(M)^\beta_\beta) \subseteq L^p_+(N)^\beta_\beta.$$

If $\alpha = 0$, then $L^p_+(N)^\alpha_\alpha$ contains only one point, and hence $\Phi(L^p_+(M)^\beta_\beta) \nsubseteq L^p_+(N)^\alpha_\alpha$. Suppose that $\alpha > 0$, and consider two distinct elements $S^{\frac{1}{p}}, T^{\frac{1}{p}} \in L^p_+(M)^\beta_\beta$ which are so close to each other that the line segment joining $S^{\frac{1}{p}}$ and $T^{\frac{1}{p}}$ lies inside $L^p_+(M)^\beta_\alpha$. Then Lemma 2.3 tells us that the line segment joining $\Phi(S^{\frac{1}{p}})$ and $\Phi(T^{\frac{1}{p}})$ lies inside $L^p_+(N)^\beta_\alpha$, which forbids both $\Phi(S^{\frac{1}{p}})$ and $\Phi(T^{\frac{1}{p}})$ belonging to $L^p_+(N)^\alpha_\alpha$ (because of the strict convexity of $L^p_{\mathrm{sa}}(N)$). This means that $\Phi(L^p_+(M)^\beta_\beta) \subseteq L^p_+(N)^\beta_\beta$. By considering Φ^{-1}, we obtain the required equality $\Phi(L^p_+(M)^\beta_\beta) = L^p_+(N)^\beta_\beta$.

Secondly, in order to establish $\Phi(L^p_+(M)^\alpha_\alpha) = L^p_+(N)^\alpha_\alpha$, it suffices to show that $\Phi(L^p_+(M)^\alpha_\alpha) \subseteq L^p_+(N)^\alpha_\alpha$ (again, thanks to the metric preserving property of Φ^{-1}). Suppose on the contrary that there exists $T^{\frac{1}{p}} \in L^p_+(M)^\alpha_\alpha$ with $\|\Phi(T^{\frac{1}{p}})\| \in (\alpha, \beta)$ (observe that $\|\Phi(T^{\frac{1}{p}})\| \neq \beta$ since $\Phi(L^p_+(M)^\beta_\beta) = L^p_+(N)^\beta_\beta$). Then $\left\|\Phi(T^{\frac{1}{p}}) - \frac{\beta\Phi(T^{\frac{1}{p}})}{\|\Phi(T^{\frac{1}{p}})\|}\right\| < \beta - \alpha$. However, for any $R^{\frac{1}{p}} \in L^p_+(M)^\beta_\beta$, one has $\|T^{\frac{1}{p}} - R^{\frac{1}{p}}\| \geq \beta - \alpha$, and this contradicts $\Phi(L^p_+(M)^\beta_\beta) = L^p_+(N)^\beta_\beta$ (as Φ preserves metric). Consequently, Relation (2.3) is verified.

Next, we define $\bar{\Phi} : L^p_+(M) \to L^p_+(N)$ by setting $\bar{\Phi}(0) = 0$ as well as

$$\bar{\Phi}(R^{\frac{1}{p}}) := \|R^{\frac{1}{p}}\|\Phi\big(\beta R^{\frac{1}{p}}/\|R^{\frac{1}{p}}\|\big)/\beta \qquad (R^{\frac{1}{p}} \in L^p_+(M) \setminus \{0\}). \tag{2.4}$$

We want to show that $\bar{\Phi}$ is a metric preserving map that extends Φ.

Indeed, if $\alpha = 0$, then by Lemma 2.3, we know that Φ is an affine map on the convex subset $L^p_+(M)^1_0$, and the requirement of $\bar{\Phi}$ follows from Lemma 2.4 (notice that $\Phi(0) = 0$ because $L^p_+(M)^0_0 = \{0\}$).

Suppose that $\alpha > 0$. Pick an arbitrary element $S^{\frac{1}{p}} \in L^p_+(M)^\beta_\beta$. It follows from

$$\|\Phi(S^{\frac{1}{p}})\| = \beta = (\beta - \alpha) + \alpha = \|\Phi(S^{\frac{1}{p}}) - \Phi(\alpha S^{\frac{1}{p}}/\beta)\| + \|\Phi(\alpha S^{\frac{1}{p}}/\beta)\|$$

and the strict convexity of $L^p_{\rm sa}(N)$ that $\Phi(S^{\frac{1}{p}}) - \Phi(\alpha S^{\frac{1}{p}}/\beta) = \delta\Phi(\alpha S^{\frac{1}{p}}/\beta)$ for some $\delta \in \mathbb{R}_+$. From this, and Relation (2.3), one has $\Phi(\alpha S^{\frac{1}{p}}/\beta) = \alpha\Phi(S^{\frac{1}{p}})/\beta$. This, together with Lemma 2.3, ensures that

$$\Phi(\gamma S^{\frac{1}{p}}) = \gamma\Phi(S^{\frac{1}{p}}) \qquad (\gamma \in [\alpha/\beta, 1]; S^{\frac{1}{p}} \in L^p_+(M)^\beta_\beta), \tag{2.5}$$

and hence $\bar{\Phi}$ extends Φ.

Consider $k \in \mathbb{Z}$. We set

$$L^p_+(M)_k := L^p_+(M)^{\beta^{k+1}/\alpha^k}_{\beta^k/\alpha^{k-1}},$$

$L^p_+(N)_k := L^p_+(N)^{\beta^{k+1}/\alpha^k}_{\beta^k/\alpha^{k-1}}$ and $\Phi_k := \bar{\Phi}|_{L^p_+(M)_k}$. It follows from (2.4) and (2.5) that

$$\Phi_k(T^{\frac{1}{p}}) = \beta^k\Phi(\alpha^k T^{\frac{1}{p}}/\beta^k)/\alpha^k \qquad (T^{\frac{1}{p}} \in L^p_+(M)_k).$$

Thus, the metric preserving property of Φ implies that Φ_k preserves metric.

Fix arbitrary distinct elements $R, T \in L^1_+(M) \setminus \{0\}$ with $\|R^{\frac{1}{p}}\| \le \|T^{\frac{1}{p}}\|$. Notice that the assignment

$$\nu : s \mapsto \|sR^{\frac{1}{p}} + (1-s)T^{\frac{1}{p}}\|$$

is a continuous map from $[0,1]$ to $\mathbb{R}_+$. There exist $k_1 \le k_2 \in \mathbb{Z}$ such that

$$\beta^{k_1}/\alpha^{k_1-1} < \|R^{\frac{1}{p}}\| \le \beta^{k_1+1}/\alpha^{k_1} \quad \text{and} \quad \beta^{k_2}/\alpha^{k_2-1} \le \|T^{\frac{1}{p}}\| < \beta^{k_2+1}/\alpha^{k_2}.$$

If $k_1 = k_2$, then $R^{\frac{1}{p}}, T^{\frac{1}{p}} \in L^p_+(M)_{k_1}$ and we have $\|\bar{\Phi}(R^{\frac{1}{p}}) - \bar{\Phi}(T^{\frac{1}{p}})\| = \|R^{\frac{1}{p}} - T^{\frac{1}{p}}\|$. Assume that $k_1 < k_2$. One can find $s_1, \dots s_n \in (0,1)$ such that $s_1 < s_2 < \cdots < s_{k_2-k_1}$ and that $\nu(s_i) = \beta^{k_1+i}/\alpha^{k_1+i-1}$. Denote

$$S_0^{\frac{1}{p}} := R^{\frac{1}{p}}, \quad S_{k_2-k_1+1}^{\frac{1}{p}} := T^{\frac{1}{p}} \quad \text{and} \quad S_i^{\frac{1}{p}} := s_i R^{\frac{1}{p}} + (1-s_i)T^{\frac{1}{p}} \quad (i = 1, \dots, k_2 - k_1).$$

Notice that $S_i^{\frac{1}{p}}, S_{i+1}^{\frac{1}{p}} \in L^p_+(M)_{k_1+i}$ $(i = 0, 1, \dots, k_2 - k_1)$, we know that

$$\|\bar{\Phi}(S_i^{\frac{1}{p}}) - \bar{\Phi}(S_{i+1}^{\frac{1}{p}})\| = \|\Phi_{k_1+i}(S_i^{\frac{1}{p}}) - \Phi_{k_1+i}(S_{i+1}^{\frac{1}{p}})\| = \|S_i^{\frac{1}{p}} - S_{i+1}^{\frac{1}{p}}\|.$$

Furthermore, since

$$\|(sR^{\frac{1}{p}} + (1-s)T^{\frac{1}{p}}) - (s'R^{\frac{1}{p}} + (1-s')T^{\frac{1}{p}})\| = (s'-s)\|R^{\frac{1}{p}} - T^{\frac{1}{p}}\| \quad \text{whenever} \quad s \le s',$$

we see that

$$\|S_0^{\frac{1}{p}} - S_1^{\frac{1}{p}}\| + \cdots + \|S_n^{\frac{1}{p}} - S_{n+1}^{\frac{1}{p}}\| = \|R^{\frac{1}{p}} - T^{\frac{1}{p}}\|.$$

Thus,

$$\|\bar{\Phi}(R^{\frac{1}{p}}) - \bar{\Phi}(T^{\frac{1}{p}})\| \le \|\bar{\Phi}(S_0^{\frac{1}{p}}) - \bar{\Phi}(S_1^{\frac{1}{p}})\| + \cdots + \|\bar{\Phi}(S_n^{\frac{1}{p}}) - \bar{\Phi}(S_{n+1}^{\frac{1}{p}})\| = \|R^{\frac{1}{p}} - T^{\frac{1}{p}}\|.$$

Furthermore, it follows the definition of $\bar{\Phi}$ that $\|\bar{\Phi}(sR^{\frac{1}{p}})\| = \|sR^{\frac{1}{p}}\|$. From these, we conclude that $\bar{\Phi}$ is contractive. By considering $\bar{\Phi}^{-1}$, we know that $\bar{\Phi} : L^p_+(M) \to L^p_+(N)$ is a metric preserving bijection extending Φ, as claimed.

Now, let us define a bijection $\Lambda : L^1_+(M)^1_1 \to L^1_+(N)^1_1$ by

$$\Lambda(S) := \big(\Phi(S^{\frac{1}{p}})\big)^p \qquad (S \in L^1_+(M)^1_1). \tag{2.6}$$

Pick arbitrary elements $R, T \in L^1_+(M)^1_1$ with $\mathbf{s}_R \cdot \mathbf{s}_T = 0$. Lemma 1.3(a) gives $\big\|R^{\frac{1}{p}} + T^{\frac{1}{p}}\big\|^p = 2$. As $\bar{\Phi}$ is metric preserving, it follows from Lemma 2.3 that

$$\|\Lambda(R)^{\frac{1}{p}} + \Lambda(T)^{\frac{1}{p}}\| = \|\bar{\Phi}(R^{\frac{1}{p}} + T^{\frac{1}{p}})\| = 2.$$

It follows again from Lemma 1.3(a) that $\mathbf{s}_{\Lambda(R)} \cdot \mathbf{s}_{\Lambda(T)} = 0$. By considering Φ^{-1}, we know that Λ is "biorthogonality preserving" in the sense of [9], and the required conclusion follows from [9, Theorem 3.2(a)]. □

The proof above can be generalized to the following statement.

Remark 2.6. Let X and Y be strictly convex normed spaces, and $K \subseteq X$ and $L \subseteq Y$ be (not necessarily proper) cones. If $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$, then a map $f : K_\alpha^\beta \to L_\alpha^\beta$ extends to a metric preserving surjection from K to L if and only if f is a metric preserving surjection.

In fact, as in the proof of Proposition 2.5, for each $k \in \mathbb{Z}$, we set $K_k := K_{\beta^k/\alpha^{k-1}}^{\beta^{k+1}/\alpha^k}$ and $L_k := L_{\beta^k/\alpha^{k-1}}^{\beta^{k+1}/\alpha^k}$. The argument of Proposition 2.5 implies that

$$f(\gamma x) = \gamma f(x) \qquad (\gamma \in [\alpha/\beta, 1]; x \in K_\beta^\beta). \tag{2.7}$$

This enable us to define a map $\bar{f} : K \setminus \{0\} \to L \setminus \{0\}$ satisfy

$$\bar{f}(x) = \beta^k f(\alpha^k x/\beta^k)/\alpha^k \qquad (x \in K_k; k \in \mathbb{Z}).$$

Furthermore, using the argument of Proposition 2.5, for every $x, y \in K \setminus \{0\}$, there exists $k_1 \leq k_2 \in \mathbb{Z}$ with $x \in K_{k_1}$ as well as $y \in K_{k_2}$, and one can find $s_0 < \cdots < s_{k_2-k_1+1}$ with $s_0 = 0$ and $s_{k_2-k_1+1} = 1$ such that $s_i x + (1 - s_i)y$ and $s_{i+1}x + (1 - s_{i+1})y$ belongs to the same K_{k_i}. From this, we know that $\bar{f}$ is metric preserving, and it extends to a metric preserving bijection from K to L if we set $\bar{f}(0) = 0$.

The above applies to the case when $K = X$ and $L = Y$. In particular, we have the following, because of the main result in [15].

Corollary 2.7. *Let $p \in (1, \infty) \setminus \{2\}$ and $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$. If $\Phi : L^p(M)_\alpha^\beta \to L^p(N)_\alpha^\beta$ is a metric preserving bijection, then there is a Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Phi(R^{\frac{1}{p}}) = \Theta_*(R)^{\frac{1}{p}}$ ($R^{\frac{1}{p}} \in L_+^p(M)_\alpha^\beta$).*

Notice that one can also use (2.7) (for $X = L^p(M) = K$ and $Y = L^p(N) = L$) as well as [3, Theorem 14.1] to get a weak conclusion as in Proposition 2.5. Note, however, that such argument cannot be applied to Proposition 2.5 in general; for example, $L_+^p([0, 1])$ cannot contain any interior point.

3. Metric preserving maps between positive annulus

In this section, we show that one can obtain a stronger conclusion than that of Theorem 3.5 in the case when M satisfies a property called EP_1, as introduced by D. Sherman in [16]. In fact, the notion of EP_p (for $p \in [1, \infty)$) in [16] is an extension of (EP) as considered by K. Watanabe in [20], which was stated in terms of $M_{*,+}$.

Definition 3.1. Let M be a von Neumann algebra.

(a) For a normed space X, a map $\chi : L_+^1(M)_1^1 \to X$ is said to be *orthogonally affine* if for every $s \in (0, 1)$,

$$\chi(sR + (1 - s)T) = s\chi(R) + (1 - s)\chi(T) \quad \text{whenever } R, T \in L_+^1(M)_1^1 \text{ satisfying } \mathbf{s}_R \cdot \mathbf{s}_T = 0.$$

(b) M is said to have EP_1 if any norm continuous orthogonally affine function $\kappa : L_+^1(M)_1^1 \to [0, 1]$ is actually affine.

Remark 3.2. (a) Our definition of EP_1 is the same as the one in [16]. In fact, suppose that $\kappa : L_+^1(M)_1^1 \to [0, 1]$ is a norm continuous orthogonally affine function. We define $\rho : L_+^1(M) \to \mathbb{R}_+$ by

$$\rho(T) := \|T\|\kappa(T/\|T\|) \qquad (T \in L_+^1(M) \setminus \{0\}).$$

Since $\|sR + (1-s)T\| = s\|R\| + (1-s)\|T\|$ for any $R, T \in L^1_+(M)$, it is not hard to check that ρ will satisfy the four conditions in [16, Definition 4.1] for $C = 1$. Conversely, if a function $\rho : L^1_+(M) \to \mathbb{R}_+$ satisfies the four conditions in [16, Definition 4.1], and we define $\kappa : L^1_+(M)^1_1 \to [0,1]$ by

$$\kappa(T) := \rho(T)/C \qquad (T \in L^1_+(M)^1_1),$$

then κ is a norm continuous orthogonally affine map.

(b) It was shown in [16, Theorem 1.2] that all semifinite algebras without type I_2 summand, all hyperfinite algebras without type I_2 summand as well as all type III_0 factors with separable preduals have EP_1. We will recall more information from [16] in the Appendix.

Lemma 3.3. *Suppose that M has EP_1. Let $\Phi : L^1_+(M)^1_1 \to L^p_+(N)^1_1$ be a norm continuous orthogonally affine map (not assumed to be surjective). Then Φ is an affine map.*

Proof. Fix an arbitrary element $f \in L^1(N)^*_+$ with $\|f\| \leq 1$. Consider the map $g : L^p_+(M)^1_1 \to [0,1]$ given by $g(R) := f(\Phi(R))$. Clearly, g is a norm-continuous orthogonally affine function. By the assumption g is affine, and hence Φ is affine (as f is arbitrary chosen). $\square$

As said in [16], the von Neumann algebra $M_2(\mathbb{C})$ does not have EP_1. In fact, Lemma 3.3 does not hold for $M = M_2(\mathbb{C})$, as shown in the following.

Example 3.4. Recall that there is a metric preserve affine bijection from $L^1_+(M_2(\mathbb{C}))^1_1$ onto the closed unit ball $\mathcal{B}$ of $\mathbb{R}^3$. The origin of $\mathcal{B}$ is the normalized trace on $M_2(\mathbb{C})$, and elements in the open unit ball are all with the same support 1. Furthermore, if $R, T \in L^1_+(M_2(\mathbb{C}))^1_1$ with $\mathbf{s}_R\mathbf{s}_T = 0$, then R and T are in the unit sphere and R is the opposite of T, i.e. the line joining R and T passes through the origin.

Now, consider a non-metric preserving homeomorphism Γ from the unit sphere $\mathcal{S}$ to itself such that whenever R is the opposite of T, then $\Gamma(R)$ is the opposite of $\Gamma(T)$. Consider $\Phi : \mathcal{B} \to \mathcal{B}$ to be the map define by the following rule: if $S = sR + (1-s)T$, where $s \in (0,1)$ where $R \in \mathcal{S}$ is the opposite of $T \in \mathcal{S}$, then $\Phi(S) = s\Gamma(R) + (1-s)\Gamma(T)$. It is easy to see that Φ is a continuous orthogonally affine map, but it cannot be affine (since continuous affine bijections between normal state spaces are defined by a Jordan *-isomorphism of the underlying algebras and hence have to be metric preserving).

Theorem 3.5. *Let $p \in (1,\infty)$, and let M and N be von Neumann algebras such that M has EP_1 and $M \ncong \mathbb{C}$. Suppose that $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$ and $\Phi : L^p_+(M)^\beta_\alpha \to L^p_+(N)^\beta_\alpha$ is a metric preserving surjection. There is a Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Phi(R^{\frac{1}{p}}) = \Theta_*(R)^{\frac{1}{p}}$ ($R^{\frac{1}{p}} \in L^p_+(M)^\beta_\alpha$).*

Proof. As in the proof of Proposition 2.5, the map Φ extends to a metric preserving affine bijection $\bar{\Phi} : L^p_+(M) \to L^p_+(N)$. Since $\bar{\Phi}(0) = 0$, we know that $\bar{\Phi}$ restricts to a bijection from $L^p_+(M)^1_1$ onto $L^p_+(N)^1_1$. Let $\Lambda : L^1_+(M)^1_1 \to L^1_+(N)^1_1$ be the bijection as defined in (2.6).

Suppose that $s \in (0,1)$ and $R, T \in L^1_+(M)^1_1$ satisfying $\mathbf{s}_R \cdot \mathbf{s}_T = 0$. It follows from Lemma 1.4(b) that

$$\begin{aligned}
\Lambda(sR + (1-s)T) &= \bar{\Phi}\big((sR + (1-s)T)^{\frac{1}{p}}\big)^p \;=\; \bar{\Phi}\big(s^{\frac{1}{p}}R^{\frac{1}{p}} + (1-s)^{\frac{1}{p}}T^{\frac{1}{p}}\big)^p \\
&= \Big((s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}})\bar{\Phi}\Big(\frac{s^{\frac{1}{p}}R^{\frac{1}{p}}}{s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}}} + \frac{(1-s)^{\frac{1}{p}}T^{\frac{1}{p}}}{s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}}}\Big)\Big)^p \\
&= \big(s^{\frac{1}{p}}\bar{\Phi}(R^{\frac{1}{p}}) + (1-s)^{\frac{1}{p}}\bar{\Phi}(T^{\frac{1}{p}})\big)^p \\
&= s\Lambda(R) + (1-s)\Lambda(T).
\end{aligned}$$

In other words, Λ is orthogonally affine.

By Lemma 1.4(a), the bijection Λ is a homeomorphism. Moreover, it follows from Lemma 3.3 that Λ is affine. Thus, [8, Theorem 4.5] gives a Jordan *-isomorphism $\Theta : N \to M$ such that for every $T \in L^1_+(M)^1_1$, one has $\Lambda(T) = \Theta_*(T)$, or equivalently, $\bar{\Phi}(T^{\frac{1}{p}}) = \Theta_*(T)^{\frac{1}{p}}$. $\square$

The above settles the last question in [10] in the case when $p \in (1, \infty)$, with the extra assumption that M has EP_1. In particular, this applies to the case when M is a semifinite algebra with no type I_2 summand and when M is a hyperfinite algebra without type I_2 summand.

The strong form as in Theorem 3.5 means that Φ is "typical", which was defined in [16] for map from $L^1_+(M)$ to $L^1_+(N)$. Since the definition for typical map does not require surjectivity, it may worth looking at the case when the map Φ is not assumed to be surjective. We will only consider the case when $\alpha = 0$ in the remark below. Notice that the main part of the extra argument required in the following remark was already given in [16]. Therefore, we do not regard it as a new result, but only state it here as an information to the readers.

Remark 3.6. Let $p \in (1, \infty)$, and let M and N be von Neumann algebras such that $M \ncong \mathbb{C}$ and has EP_1. Suppose that $\Psi : L^p_+(M)^1_0 \to L^p_+(N)^1_0$ is a metric preserving map (not assume to be surjective) such that $\Psi(0) = 0$. Then Ψ is typical in the sense of [16]

In fact, by Lemmas 2.3 and 2.4, we know that Ψ extends to an affine metric preserving map $\bar{\Psi} : L^p_+(M) \to L^p_+(N)$ (not necessarily surjective). Let $\Lambda : L^1_+(M)^1_1 \to L^1_+(N)^1_1$ be the (not necessarily surjective) map defined in a similar way as (2.6). Then the argument of Theorem 3.5 tells us that Λ is orthogonally affine, and Lemma 3.3 gives the affineness of Λ. Furthermore, we define $\tilde{\Lambda} : L^1_{\rm sa}(M) \to L^1_{\rm sa}(N)$ by $\tilde{\Lambda}(T) = \bar{\Lambda}(T_+) - \bar{\Lambda}(T_-)$, where $\bar{\Lambda}(S) := \|S\|\Lambda(S/\|S\|)$ when $S \neq 0$. For any $y \in N_{\rm sa}$, the function $y \circ \Lambda$ is continuous and affine on $L^1_+(M)^1_1$ and hence there exists $\Lambda^*(y) \in M_{\rm sa}$ such that

$$R(\Lambda^*(y)) = \Lambda(R)(y) \qquad (R \in L^1_+(M)^1_1)$$

(see e.g. [1, Theorem 11.5]), which gives $\tilde{\Lambda}(T)(y) = T(\Lambda^*(y))$ $(T \in L^1_{\rm sa}(M))$. Consequently, $\tilde{\Lambda}$ is real linear and extends to a bounded complex linear map, again denoted by $\tilde{\Lambda}$, from $L^1(M)$ to $L^1(N)$. Moreover, one can use Lemma 1.3(a) to show that Λ is orthogonality preserving (see (1.1)), and hence $\tilde{\Lambda}$ is an "o.d. homomorphism" in the sense of [4]. Now, it follows from the argument in the last two paragraphs preceding [16, Theorem 4.3] that Ψ is typical.

Appendix A. Algebras with EP_1

In [16, Theorem 1.2], some algebras with EP_1 were listed, and their proofs were given in the main body of [16] (in fact, the more general case of EP_p was considered there). In particular, it was shown that approximately semifinite algebra with no type I_2 summand has EP_1. However, the proof for this fact seems to scatter in [16] and is not easy to trace. For the benefit of the readers, we collect some facts from [16] that leads to the above statement. There is no new result nor new proof given in this appendix.

First of all, one can find in [16, Theorem 5.3] and its proof the following lemma.

Lemma A.1. *Let M be a von Neumann algebras.*

(a) If M is finite and has no type I_2 summand, then M has EP_1.

(b) If there is an increasing net $\{M_i\}_{i\in\mathfrak{I}}$ of von Neumann subalgebras (of M) having EP_1 with $\bigcup_{i\in\mathfrak{I}} M_i$ being $\sigma(M, M_*)$-dense in M, and for each $i \in \mathfrak{I}$, there is a normal conditional expectation $E_i : M \to M_i$ such that $E_i(1)$ is the identity of M_i and that $E_i \circ E_j = E_i$ whenever $i \leq j$, then M has EP_1.

Suppose now that M is a semifinite algebra without type I_2 summand. Let M_1 and M_2 be the type I and the type II parts of M respectively. Clearly, qM_2q does not have any type I_2 summand, for any $q \in \mathcal{P}(M_2)$. On the other hand, M_1 can be decomposed as $\bigoplus_{\alpha\in\Lambda} L^\infty(X_\alpha, \mathcal{L}(\mathfrak{H}_\alpha))$ with $\dim \mathfrak{H}_\alpha \neq 2$ for every $\alpha \in \Lambda$. Thus, there exists an increasing net $\{p_i\}_{i\in\mathfrak{I}}$ in the set

$$\{p \in \mathcal{P}(M) : pMp \text{ has a faithful tracial state and does not have any type } I_2 \text{ summand}\}$$

that $\sigma(M, M_*)$-converges to 1. This, together with Lemma A.1, gives [16, Theorem 5.3(b)], which we recall in the following.

Proposition A.2. *If M is a semifinite von Neumann algebra with no type I_2 summand, then M has EP_1.*

Our next lemma follows readily from the definition of EP_1, because all elements in $L^1_+(M)^1_1$ have disjoint supports from elements in $L^1_+(N)^1_1$.

Lemma A.3. *If M and N are two von Neumann algebras with EP_1, then $M \oplus N$ has EP_1.*

Let us now recall the definition of approximately semifinite algebras.

Definition A.4. A von Neumann algebra M is said to be *approximately semifinite* if there is a net $\{E_i\}_{i\in\mathfrak{I}}$ of normal conditional expectations from M onto an increasing net $\{M_i\}_{i\in\mathfrak{I}}$ of semifinite von Neumann subalgebras, with $E_i \circ E_j = E_i$ and $E_i(1)$ being the identity of M_i for any $i \leq j$ in $\mathfrak{I}$, such that $\bigcup_{i\in\mathfrak{I}} M_i$ is $\sigma(M, M_*)$-dense in M. In this case, $\{(M_i, E_i)\}_{i\in\mathfrak{I}}$ is called a *semifinite paving* for M.

The following fact is also clear. Indeed, if $\{(M_i, E_i)\}_{i\in\mathfrak{I}}$ is a semifinite paving for M, and $P : M \to N$ is the canonical projection, then $\{(P(M_i), P \circ E_i|_N)\}_{i\in\mathfrak{I}}$ is a semifinite paving for N.

Lemma A.5. *Suppose that M is approximately semifinite. If $M = L \oplus N$, then N is also approximately semifinite.*

Proposition A.6. *If M is an approximately semifinite von Neumann algebra with no type I_2 summand, then M has EP_1.*

In fact, we consider L and N to be the finite part and the properly infinite part of M, respectively. It follows from Lemma A.1(a) that L has EP_1. Moreover, by Lemma A.5, the algebra N is approximately semifinite. If $\{(N_i, E_i)\}_{i\in\mathfrak{I}}$ is a semifinite paving for N, then $\{(N_i \otimes M_3(\mathbb{C}), E_i \otimes \mathrm{id})\}_{i\in\mathfrak{I}}$ is a semifinite paving for $N \otimes M_3(\mathbb{C}) \cong N$ (because N is properly infinite). Since the semifinite algebra $N_i \otimes M_3(\mathbb{C})$ can never have a type I_2 summand, we know from Proposition A.2 and Lemma A.1(b) that N has EP_1. Now, it follows from Lemma A.3 that M has EP_1.

Acknowledgement

The authors are supported by National Natural Science Foundation of China (11471168) and Taiwan MOST grant (104-2115-M-110-009-MY2). The authors would also like to thanks K. Watanabe for some discussion relating to the propery (EP) and for informing them [16] and [20].

References

[1] E.M. Alfsen and F.W. Shultz, *Non-commutative spectral theory for affine function spaces on convex sets*, Memoirs Amer. Math. Soc. **172**, Providence (1976).
[2] J.A. Baker, Isometries in normed spaces, Amer. Math. Monthly, **78** (1971), 655-658.
[3] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Amer. Math. Soc. Collo. Publ. **48**, Amer. Math. Soc. (2000).
[4] L.J. Bunce and J.D.M. Wright, On orthomorphism between von Neumann preduals and a problem of Araki, Pac. J. Math. **158** (1993), 265-272.
[5] A. Connes, Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, Ann. Inst. Fourier **24** (1974), 121-155.
[6] H.A. Dye, On the geometry of projections in certain operator algebras, Ann. Math. **61** (1955), 73-89.
[7] R. V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. Math. **564** (1952), 494-503.
[8] R. V. Kadison, Transformations of states in operator theory and dynamics, Topology **3** (1965) suppl. 2, 177-198.

[9] C.-W. Leung, C.-K. Ng and N.-C. Wong, Transition probabilities of normal states determine the Jordan structure of a quantum system, J. Math. Phys. **57** (2016), 015212, 13 pages; doi: 10.1063/1.4936404.
[10] C.-W. Leung, C.-K. Ng, N.-C. Wong, The positive contractive part of a Noncommutative L^p-space is a complete Jordan invariant, Lin. Alg. Appl., to appear.
[11] G. Pisier and Q. Xu, Non-commutative Lp-spaces, *Handbook of the geometry of Banach spaces, Vol. 2*, North-Holland, Amsterdam (2003), 1459-1517.
[12] Y. Raynaud and Q. Xu, On subspaces of noncommutative L_p-spaces, J. Funct. Anal. **203** (2003), 149-196.
[13] É. Ricard, Hölder estimates for the noncommutative Mazur maps, Arch. Math. **104**, (2015), 37-45.
[14] B. Russo, Isometrics of L^p-spaces associated with finite von Neumann algebras, Bull. Amer. Math. Soc. **74** (1968), 228-232.
[15] D. Sherman, Noncommutative L^p-structure encodes exactly Jordan structure, J. Funct. Anal. **211** (2005), 150-166.
[16] D. Sherman, On the Structure of Isometries between Noncommutative L^p Spaces, Publ. RIMS Kyoto Univ., **42** (2006), 45-82.
[17] M. Takesaki, *Theory of operator algebras II*, Encyclopaedia of mathematical sciences **125** (2003), Springer-Verlag.
[18] P.-K. Tam, Isometries of L_p-spaces associated with semifinite von Neumann algebras, Trans. Amer. Math. Soc. **254** (1979), 339-354.
[19] M. Terp, L^p-spaces associated with von Neumann algebras, Notes Math. Institute, Copenhagen Univ. (1981).
[20] K. Watanabe, Problems on isometries of non-commutative Lp-spaces, Contemp. Math. **232**, Amer. Math. Soc. (1999), 349-356.
[21] F.J. Yeadon, Isometries of noncommutative L^p-spaces. Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 1, 41-50.
[22] F.J. Yeadon, Finitely additive measures on projections in finite W^*-algebras, Bull. Lond. Math. Soc. **16** (1984), 145-150.

(Chi-Wai Leung) Department of Mathematics, The Chinese University of Hong Kong, Hong Kong.

E-mail address: cwleung@math.cuhk.edu.hk

(Chi-Keung Ng) Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China.

E-mail address: ckng@nankai.edu.cn

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw