

THE SPECTRUM OF THE PRODUCT OF OPERATORS, AND THE PRODUCT OF THEIR NUMERICAL RANGES

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This paper is dedicated to Professor Pei Yuan Wu

ABSTRACT. We show that a compact operator A is a multiple of a positive semi-definite operator if and only if

$$\sigma(AB) \subseteq \overline{W(A)W(B)}, \quad \text{for all (rank one) operators } B.$$

An example of a normal operator is given to show that the equivalence conditions may fail in general. We then obtain conditions to identify other classes of operators A so that equivalence conditions hold.

1. INTRODUCTION

Let $B(H)$ be the algebra of bounded linear operators on a complex Hilbert space H . We identify $B(H)$ with M_n , the algebra of $n \times n$ complex matrices, if H has finite dimension n . The *spectrum* $\sigma(A)$, and the *numerical range* $W(A)$ of an operator $A \in B(H)$ are defined by

$$\sigma(A) = \{\lambda : A - \lambda I \text{ is not invertible}\}, \quad \text{and} \quad W(A) = \{\langle Ax, x \rangle : x \in H, \|x\| = 1\},$$

respectively. Here, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and its corresponding norm of H .

The Hausdorff–Toeplitz theorem asserts that $W(A)$ is always a bounded convex subset of the complex plane. When H is finite dimensional, it is compact. In general, the closure of the numerical range satisfies

$$\sigma(A) \subseteq \overline{W(A)}.$$

When A is normal, we have

$$\text{conv } \sigma(A) = \overline{W(A)}.$$

Here, $\text{conv } S$ denotes the convex hull of a set S in a vector space. The spectrum and the numerical range are useful tools for studying operators and matrices. Motivated by the theoretical development and applications, researchers have obtained many interesting results; see, for example, [8], [9, Chapter 22] or [11, Chapter 1].

In perturbation theory, one might want to estimate $\sigma(A+B)$ for “small” B , but it is known that $\sigma(A+B) \not\subseteq \sigma(A) + \sigma(B)$. For example, let $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$ with positive $M, \varepsilon > 0$.

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Then $\sigma(A) = \sigma(B) = \{0\}$, whereas $\sigma(A + B) = \{\pm\sqrt{M\varepsilon}\}$. Nevertheless, we always have

$$\sigma(A + B) \subseteq W(A + B) \subseteq W(A) + W(B).$$

Thus, $W(A) + W(B)$ provides a containment region for $\sigma(A + B)$.

In (multiplicative) perturbation theory, one considers $\tilde{A} = AB$ such that B is closed to the identity operator I . However, neither of the inclusion $\sigma(AB) \subseteq \sigma(A)\sigma(B)$ nor $W(AB) \subseteq W(A)W(B)$ holds. The following example in [10] tells us that the above inclusions may not hold even for 2×2 hermitian matrices A, B . Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\sigma(AB) = \{\pm\sqrt{-1}\} \not\subseteq W(A)W(B) = [-1, 1].$$

Nevertheless, it was shown in [13] that if $A, B \in B(H)$ and $0 \notin \overline{W(A)}$, then

$$\sigma(A^{-1}B) \subseteq \overline{W(B)}/\overline{W(A)}.$$

It follows from this result that if $A \in M_n$ is a multiple of a positive semi-definite matrix,

$$\sigma(AB) \subseteq W(A)W(B) \quad \text{for all } B \in M_n.$$

In [5, Theorem 3], it was shown that if $A \in B(H)$ is a (multiple of a) positive semi-definite operator, then

$$\text{conv } \sigma(AB) \subseteq \overline{W(A)W(B)} \quad \text{for all } B \in B(H).$$

It is natural to consider the converse problem; namely,

Question 1.1. *Is $A \in B(H)$ a multiple of a positive semi-definite matrix if*

$$\sigma(AB) \subseteq \overline{W(A)W(B)} \quad \text{for all } B \in B(H)?$$

In [1], the author considered this question for matrices $A \in M_n$, and an affirmative answer to this question was claimed in the paper. In [6, 7], the authors there pointed out some gaps in the argument in [1], and repaired them.

In this paper, we consider Question 1.1 for infinite dimensional operators. In particular, we refine the finite dimensional result to the following.

Theorem 1.2. *Suppose $A \in B(H)$ is a compact operator. The following conditions are equivalent.*

- (A1) *A is a multiple of a positive semi-definite operator.*
- (A2) $\sigma(AB) \subseteq \overline{W(A)W(B)}$ for all $B \in B(H)$.
- (A3) $\sigma(AB) \subseteq \overline{W(A)W(B)}$ for all rank one $B \in B(H)$.

Of course, it would be nice to further extend the result to general operators. However, the following example shows that it is impossible even on a separable Hilbert space. (A verification of the example will be given in the next section).

Example 1.3. Consider a Hilbert space H with a countable orthonormal basis $\{f_1, f_2, \dots\}$. Suppose $\{\mu_1, \mu_2, \dots\}$ is a dense set of the unit circle $\{e^{it} : t \in [0, 2\pi)\}$ and T is the diagonal operator on H satisfying $Tf_n = \mu_n f_n$. Then $A := I + T$, which is not a multiple of positive semi-definite operator, satisfies

$$\sigma(AB) \subseteq \overline{W(A)W(B)}, \quad \text{for all } B \in B(H).$$

A key step in the proof of the finite dimensional result is to show that if $A \in M_n$ satisfies condition **(A2)** in Theorem 1.2, then A is normal. We can modify Example 1.3 to show that this implication is not true for general operators in the following. (The verification will also be done in the next section.)

Example 1.4. Suppose $\hat{A} = A \oplus \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \in B(H \oplus \mathbf{C}^2)$, where A is defined as in Example 1.3 and $d \in (0, 1]$. Then \hat{A} is not normal and condition **(A2)** in Theorem 1.2 holds.

Apart from Examples 1.3 and 1.4, we obtain the following theorem, which allows us to identify other classes of operators A such that the conditions **(A1)**, **(A2)**, **(A3)** are equivalent.

Theorem 1.5. Let H be a Hilbert space of finite or infinite dimension. Consider the following conditions for an operator $A \in B(H)$.

- (A1)** A is a multiple of a positive (semi-definite) operator.
- (A2)** $\sigma(AB) \subseteq \overline{W(A)W(B)}$ for all $B \in B(H)$.
- (A3)** $\sigma(AB) \subseteq \overline{W(A)W(B)}$ for all rank one $B \in B(H)$.

Then the following implications hold:

$$\mathbf{(A1)} \Rightarrow \mathbf{(A2)} \Rightarrow \mathbf{(A3)}.$$

The implication

$$\mathbf{(A3)} \Rightarrow \mathbf{(A1)}$$

holds when there is a boundary point μ of $\overline{W(A)}$ attaining the numerical radius $|\mu| = w(A)$ and lying on two different support lines of $\overline{W(A)}$.

By Theorem 1.5, we have the following.

Corollary 1.6. In each of the following cases, conditions **(A1)**, **(A2)**, **(A3)** in Theorem 1.5 are equivalent for an operator $A \in B(H)$.

- (1) $\overline{W(A)}$ is a convex polygon, which may degenerate to a line segment or a point. This covers the cases when A is a scalar multiple of a hermitian operator, or when A is a normal operator with finite spectrum.
- (2) $A \in B(H)$ is normal and there is an isolated point λ in $\sigma(A)$ attaining the spectral radius $|\lambda| = r(A)$.

2. PROOFS AND AUXILIARY RESULTS

We focus on the proof of Theorem 1.5, and deduce Theorem 1.2 and Corollary 1.6 as consequences. We then verify Examples 1.3 and 1.4.

The implication **(A1)** \Rightarrow **(A2)** in Theorem 1.5 is a result in [5]. Furthermore, it was shown that if **(A1)** holds, then $\overline{W(A)W(B)}$ is always convex. We give a short proof of the result.

Proposition 2.1. *Suppose $A \in B(H)$ is a multiple of a positive semi-definite operator. Then for any $B \in B(H)$, the set $\overline{W(A)W(B)}$ is convex, and*

$$\sigma(AB) \subseteq \overline{W(A)W(B)}.$$

Proof. Without loss of generality, we can assume that A is positive semi-definite. Note that $\overline{W(A)} = [a_1, a_2]$, $a_2 \geq a_1 \geq 0$, and $\overline{W(B)}$ is a compact convex set. Thus,

$$\overline{W(A)W(B)} = \bigcup_{\mu \in \overline{W(A)}} \mu \overline{W(B)} = \bigcup_{a_1 \leq t \leq a_2} t \overline{W(B)}$$

is convex.

Now, suppose $\lambda \in \sigma(AB)$. If $\lambda = 0$, then AB is singular, so that A is singular or B is singular. Hence, $0 \in \sigma(A) \in \overline{W(A)}$ or $0 \in \sigma(B) \in \overline{W(B)}$ so that $0 \in \overline{W(A)W(B)}$.

If $\lambda \neq 0$, then $\lambda \in \sigma(AB) \setminus \{0\} = \sigma(A^{1/2}BA^{1/2}) \setminus \{0\}$.

Assume first that λ is an approximate eigenvalue of $A^{1/2}BA^{1/2}$. Then there is a sequence of unit vectors $\{x_n\}$ such that

$$\|A^{1/2}BA^{1/2}x_n - \lambda x_n\| \rightarrow 0.$$

We may assume that $A^{1/2}x_n \neq 0$ for all $n \in \mathbb{N}$. Thus $t_n \langle Bx_n, x_n \rangle \rightarrow \lambda$ in $\overline{W(A)W(B)}$ with

$$y_n = A^{1/2}x_n / \|A^{1/2}x_n\| \text{ and } t_n = \langle Ax_n, x_n \rangle \in W(A).$$

Assume next that λ is not in the approximate point spectrum, and thus $A^{1/2}BA^{1/2} - \lambda$ does not have a dense range. Consequently, we can find a norm one element y in H orthogonal to its range. In particular,

$$\langle (A^{1/2}BA^{1/2} - \lambda)y, y \rangle = 0.$$

This gives

$$\lambda = \langle A^{1/2}BA^{1/2}y, y \rangle = t \langle Bx, x \rangle,$$

and $A^{1/2}y \neq 0$. Here, $t = \|A^{1/2}y\|^2 = \langle Ay, y \rangle \in W(A)$, and $x = A^{1/2}y / \|A^{1/2}y\|$ is of norm one. Hence, $\lambda \in W(A)W(B)$. ■

The implication **(A2)** \Rightarrow **(A3)** is clear. We now focus on the condition under which the implication **(A3)** \Rightarrow **(A1)** holds.

Proposition 2.2. *Suppose $A \in B(H)$ satisfies **(A3)**. Then there is $\mu \in \overline{W(A)}$ such that $|\mu| = w(A)$. Moreover, if such a μ lies on two different support lines of $\overline{W(A)}$, then **(A1)** holds.*

We need some preliminaries to prove Proposition 2.2. Let $A \in B(H)$ satisfying **(A3)**. Note that conditions **(A1)**, **(A2)** and **(A3)** will not be affected by replacing A with $\gamma UA^\ddagger U^*$ for any nonzero γ , unitary $U \in B(H)$, $A^\ddagger \in \{A, A^t, A^*\}$. We will use this fact in our proof.

Lemma 2.3. *Suppose $A \in B(H)$ satisfies **(A3)**.*

- (1) *The operator A is radialoid. That is, $r(A) = w(A) = \|A\|$.*
(2) *Suppose $\eta \in W(A)$ satisfies $|\eta| = w(A)$. Then $|\eta - \mu| \leq w(A)$ for all $\mu \in W(A)$.*

Proof. We may replace A by $A/\|A\|$ and assume that $\|A\| = 1$. To prove (1), suppose $Ax_n = \lambda_n y_n$ for some unit vectors $x_n, y_n \in H$ with positive scalars $\lambda_n \uparrow 1$. Let $B_n \in B(H)$ be the rank one operator $z \mapsto \langle z, y_n \rangle x_n$. Then $AB_n y_n = \lambda_n y_n$, and thus $\lambda_n \in \sigma(AB_n) \subseteq \overline{W(A)W(B_n)}$. Since $w(B_n) \leq 1$, we have $\lambda_n \leq w(A)$. That is, $1 \leq w(A)$. Thus $w(A) = 1$, and hence $r(A) = 1$ (see [8, Theorem 1.3-2]).

Next, consider (2). It is well known that if $w(A) = \|A\|$ and $\eta \in W(A)$ satisfies $|\eta| = w(A)$, then A is unitarily similar to $\eta I \oplus A_1$. We may replace A by $A/|\eta|$ and suppose $\eta \in W(A)$ with $|\eta| = 1$. Then for any unit vector x such that $\langle Ax, x \rangle = \eta$, we write $Ax = \eta x + \nu y$ for some unit vector y orthogonal to x . Then

$$|\eta|^2 = \|A\|^2 \geq \|Ax\|^2 = |\eta|^2 + |\nu|^2.$$

Thus, $\nu = 0$ and $Ax = \eta x$. The first assertion follows.

For the second assertion, we may replace A by A/η and assume that $A = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}$. Let $\lambda \in W(A_1)$. We can assume that the leading (upper left) 2×2 submatrix of A is $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$. Let $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ be the 2×2 unitary matrix. Then the leading 2×2 submatrix of $\hat{A} = (U \oplus I)A(U \oplus I)^*$ equals $A_0 = \frac{1}{2} \begin{pmatrix} 1 + \lambda & 1 - \lambda \\ 1 - \lambda & 1 + \lambda \end{pmatrix}$. Let $B = (U \oplus I)^* \left(\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \oplus O \right) (U \oplus I)$. Then $\text{tr}(AB) = 1 - \lambda$ is the nonzero eigenvalue of the rank one matrix AB . Since $\sigma(AB) \subseteq \overline{W(A)W(B)} \subseteq \{z \in \mathbf{C} : |z| \leq 1\}$, we have $|1 - \lambda| \leq 1$.

The last assertion follows from the fact that $W(A) = \text{conv}(\{1\} \cup W(A_1))$. ■

In [1], the authors showed that if a matrix A satisfies **(A2)**, then there is $\mu \in W(A)$ satisfying $|\mu| = \|A\|$, and tried to prove that **(A1)** holds. Lemma 2.3(1) shows that for any $A \in B(H)$ satisfying **(A3)**, there is $\mu \in \overline{W(A)}$ such that $|\mu| = \|A\|$ and μ is an eigenvalue of A .

We will use Lemma 2.3 to prove Proposition 2.2. In the finite dimensional case, $W(A) = \overline{W(A)}$ is compact, and there are unit vectors attaining the norm of A . However, it might not be the case if the underlying Hilbert space H is infinite dimensional. Nevertheless, we can use the Berberian construction (see [2]) to overcome this technicality.

In connection to our problem, we will impose additional requirement in the Berberian construction, namely, we will need a generalized Banach limit which is multiplicative. We include some details of the construction for completeness. We identify the space ℓ_∞ of bounded scalar sequences with the C^* -algebra $C(\beta\mathbb{N})$ of continuous functions on the Stone-Cech compactification $\beta\mathbb{N}$ of \mathbb{N} . Here, a bounded sequence $\lambda = (\lambda_n)$ in ℓ_∞ corresponds to a function $\hat{\lambda}$ in $C(\beta\mathbb{N})$ with $\hat{\lambda}(n) = \lambda_n$ for all $n = 1, 2, \dots$. Take any point ξ from $\beta\mathbb{N} \setminus \mathbb{N}$. The point evaluation $\lambda \mapsto \hat{\lambda}(\xi)$ of ℓ_∞ gives a nonzero multiplicative generalized Banach limit, denoted by glim , that satisfies the following conditions. For any bounded sequences (a_n) and (b_n) in ℓ_∞ and scalar γ , we have

- (a) $\text{glim}(a_n + b_n) = \text{glim}(a_n) + \text{glim}(b_n)$.
- (b) $\text{glim}(\gamma a_n) = \gamma \text{glim}(a_n)$.
- (c) $\text{glim}(a_n) = \lim a_n$ whenever $\lim a_n$ exists.
- (d) $\text{glim}(a_n) \geq 0$ whenever $a_n \geq 0$ for all n .
- (e) $\text{glim}(a_n b_n) = \text{glim}(a_n) \text{glim}(b_n)$.

Equivalently, we can define $\text{glim}(a_n) = \lim_{\mathfrak{U}} a_n$ through a free ultrafilter \mathfrak{U} on \mathbb{N} , when we consider $\beta\mathbb{N}$ consisting of ultrafilters on \mathbb{N} and those outside \mathbb{N} are free (i.e. $\bigcap \mathfrak{U} = \emptyset$). Note that all multiplicative generalized Banach limits on ℓ_∞ arise from the above construction. Note also that we do not assume the translation invariant property on glim . Indeed, the only translation invariant multiplicative generalized Banach limit is zero.

Denote by \mathcal{V} the set of all bounded sequences $\{x_n\}$ with $x_n \in H$. Then \mathcal{V} is a vector space relative to the definitions $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\gamma\{x_n\} = \{\gamma x_n\}$. Let \mathcal{N} be the set of all sequences $\{x_n\}$ such that $\text{glim}(\langle x_n, x_n \rangle) = 0$. Then \mathcal{N} is a linear subspace of \mathcal{V} . Denote by \mathbf{x} the coset $\{x_n\} + \mathcal{N}$. The quotient vector space \mathcal{V}/\mathcal{N} becomes an inner product space with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \text{glim}(\langle x_n, y_n \rangle)$. Let K be the completion of \mathcal{V}/\mathcal{N} . If $x \in H$, then $\{x\}$ denotes the constant sequence defined by x . Since $\langle \mathbf{x}, \mathbf{y} \rangle = \langle x, y \rangle$ for $\mathbf{x} = \{x\} + \mathcal{N}$ and $\mathbf{y} = \{y\} + \mathcal{N}$, the mapping $x \mapsto \mathbf{x}$ is an isometric linear map of H onto a closed subspace of K and K is an extension of H . For an operator $A \in \mathcal{B}(H)$, define

$$A_0(\{x_n\} + \mathcal{N}) = \{Ax_n\} + \mathcal{N}.$$

We can extend A_0 on K , which will be denoted by A_0 also. The mapping $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ given by $\phi(T) = \tilde{T}$ is a unital isometric $*$ -representation with $\sigma(T) = \sigma(\tilde{T})$. Moreover, the approximate eigenvalues of T (and also \tilde{T}) will become eigenvalues of \tilde{T} . See [2].

It is clear that rank one operators in $B(H)$ become rank one operators in $B(K)$. However, rank one operators in $B(K)$ does not necessarily come from rank one operators in $B(H)$. A counter example can be given by the rank one operator $e \otimes e$ defined by $k \mapsto \langle k, e \rangle e$ for a nonzero vector e in K orthogonal to H . Nevertheless, in connection to our study, we have the following.

Lemma 2.4. *Let $\tilde{A} \in B(K)$ be the extension of $A \in B(H)$ in the Berberian construction. Suppose $\sigma(AB) \subseteq \overline{W(A)W(B)}$ for all rank one $B \in B(H)$. Then*

$$\sigma(\tilde{A}B') \subseteq \overline{W(\tilde{A})W(B')} \text{ for all rank one } B' \in B(K).$$

Proof. To prove our assertion, we make some observations.

- (i) $W(\tilde{A}) = \overline{W(A)}$ for all $A \in B(H)$ by [3].
- (ii) $W(B) = \overline{W(B)}$ for any rank one $B \in B(H)$.
- (iii) If $\mathbf{x}, \mathbf{y}, \mathbf{u} \in K$ correspond to the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ of unit vectors in H , then

$$\langle (\mathbf{x} \otimes \mathbf{y})\mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{y} \rangle \langle \mathbf{x}, \mathbf{u} \rangle = \text{glim}(\langle u_n, y_n \rangle) \text{glim}(\langle x_n, u_n \rangle) = \text{glim}[\langle u_n, y_n \rangle \langle x_n, u_n \rangle].$$

Let $B' = \mathbf{x} \otimes \mathbf{y}$ with \mathbf{x}, \mathbf{y} in K arising from the sequences $\{x_n\}, \{y_n\}$ of unit vectors in H . We will show that

$$\sigma(\tilde{A}B') \subseteq \{ \langle \tilde{A}\mathbf{x}, \mathbf{y} \rangle, 0 \} \subseteq \overline{W(\tilde{A})W(B')}.$$

Obviously, $0 \in W(\tilde{A})W(B')$. Observe that

$$\langle \tilde{A}\mathbf{x}, \mathbf{y} \rangle = \text{glim} (\langle Ax_n, y_n \rangle),$$

where

$$\langle Ax_n, y_n \rangle \in \sigma(A(x_n \otimes y_n)) \subseteq \overline{W(A)W(x_n \otimes y_n)} = W(\tilde{A})W(x_n \otimes y_n)$$

by (i) and (ii) above. It follows that

$$\langle Ax_n, y_n \rangle = \langle \tilde{A}v_n, v_n \rangle \langle (x_n \otimes y_n)u_n, u_n \rangle = \langle \tilde{A}v_n, v_n \rangle \langle u_n, y_n \rangle \langle x_n, u_n \rangle$$

for some unit vector $v_n \in K$ and unit vector $u_n \in H$. By (iii) above,

$$\text{glim} [\langle u_n, y_n \rangle \langle x_n, u_n \rangle] = \langle (\mathbf{x} \otimes \mathbf{y})\mathbf{u}, \mathbf{u} \rangle \in W(\mathbf{x} \otimes \mathbf{y}) = W(B').$$

By the compactness of $W(\tilde{A})$ and $W(B')$, we have

$$\text{glim} \langle Ax_n, y_n \rangle = \text{glim} (\langle \tilde{A}v_n, v_n \rangle) \text{glim} [\langle u_n, y_n \rangle \langle x_n, u_n \rangle] \in W(\tilde{A})W(B'). \quad \blacksquare$$

Proof of Proposition 2.2. We may replace A by $A/\|A\|$ and assume that $\|A\| = 1$. Furthermore, we may apply the Berberian construction and assume that conditions (a) – (e) hold. For simplicity, we assume $H = K$ and $A = \tilde{A}$.

By Lemma 2.3, we have $r(A) = w(A) = \|A\| = 1$ and we may assume that $A = \mu I \oplus A_1$ for some contraction A_1 such that μ is not an eigenvalue of A_1 . Without loss of generality, we may assume that $\mu = 1$, $A = I \oplus A_1$. We need to show that A_1 is positive semi-definite. Assume that it is not the case so that $W(A_1) \not\subseteq [0, 1]$.

By our assumption, there is a support line of $W(A)$ passing through 1 and $1 + r_1 e^{i\alpha_1}$ for some $\pi/2 < \alpha_1 < 3\pi/2$ and $r_1 \in [0, 1]$. Replacing A with A^* if necessary, we can also assume that $\pi/2 < \alpha_1 < \pi$ and

$$W(A) \subseteq \{z = 1 + r e^{i\alpha} : r \in [0, 1], \alpha_1 \leq \alpha \leq 3\pi/2, |z| \leq 1\}.$$

Let $B = B_0 \oplus O$ with

$$B_0 = 2e^{i(\pi/2 - \alpha_1)} \begin{pmatrix} \cos \theta & 0 \\ \sin \theta & 0 \end{pmatrix},$$

where $\theta \in (0, \pi/2)$ such that $2r_1 \sin \alpha_1 \geq \tan \theta$. Observe that

$$\begin{aligned} (\sin \alpha_1 \cos \theta + r_1 \sin \theta)^2 &= (\sin \alpha_1 \cos \theta)^2 + 2r_1 \sin \alpha_1 \cos \theta \sin \theta + r_1^2 \sin^2 \theta \\ &> (\sin \alpha_1 \cos \theta)^2 + \sin^2 \theta, \end{aligned}$$

i.e.,

$$(2.1) \quad \sin \alpha_1 \cos \theta + r_1 \sin \theta > \sqrt{(\sin \alpha_1 \cos \theta)^2 + \sin^2 \theta}.$$

With a suitable unitary transform, we may assume that A has a leading 2×2 submatrix $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 + r_1 e^{i\alpha_1} \end{pmatrix}$. Let $U = U_0 \oplus I$ with $U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{aligned} \lambda = \operatorname{tr}(UAU^*B) &= \operatorname{tr}(U_0 A_0 U_0^* B_0) \\ &= 2e^{i(\pi/2 - \alpha_1)} \cos \theta + r_1 e^{i\alpha_1} \operatorname{tr}(U_0 E_{22} U_0^* B_0) \\ &= 2e^{i(\pi/2 - \alpha_1)} \cos \theta + r_1 e^{i\alpha_1} \operatorname{tr}(E_{22} U_0^* B_0 U_0) \\ &= 2 \sin \alpha_1 \cos \theta + r_1 \sin \theta + i(2 \cos \alpha_1 \cos \theta + r_1 \cos \theta), \end{aligned}$$

which is the nonzero eigenvalue of the rank one matrix UAU^*B .

To derive a contradiction, we will show that

$$(2.2) \quad \lambda \notin W(A)W(B) = \bigcup_{z \in W(A)} W(zB).$$

Recall that for any compact operator, and thus any finite matrix, T , the right support line of $W(T)$ is the set of complex numbers with real part equal to the maximum eigenvalue of $(T + T^*)/2$. For each $z = 1 + r e^{i\alpha} \in W(A)$ with $r \in [0, 1]$ and $\alpha_1 \leq \alpha < 3\pi/2$, the maximum eigenvalue of the matrix

$$\frac{1}{2}(zB_0 + \bar{z}B_0^*) = \begin{pmatrix} (\sin \alpha_1 + r \sin(\alpha_1 - \alpha))2 \cos \theta & -i \cdot e^{i\alpha_1} \cdot \bar{z} \sin \theta \\ i \cdot e^{-i\alpha_1} \cdot z \sin \theta & 0 \end{pmatrix}$$

equals

$$(2.3) \quad \gamma + \sqrt{\gamma^2 + |z|^2 \sin^2 \theta} \quad \text{with } \gamma = (\sin \alpha_1 + r \sin(\alpha_1 - \alpha)) \cos \theta.$$

Because $\pi/2 < \alpha_1 \leq \alpha < 3\pi/2$, we have $\alpha_1 - \alpha \in (-\pi, 0)$ and $\sin(\alpha_1 - \alpha) \leq 0$. Suppose $\gamma \geq 0$, i.e., $\sin \alpha_1 \geq |r \sin(\alpha_1 - \alpha)|$. Then by (2.1) we have

$$\gamma + \sqrt{\gamma^2 + |z|^2 \sin^2 \theta} \leq \sin \alpha_1 \cos \theta + \sqrt{(\sin \alpha_1 \cos \theta)^2 + \sin^2 \theta} < 2 \sin \alpha_1 \cos \theta + r_1 \sin \theta.$$

If $\gamma < 0$, i.e., $\sin \alpha_1 < |r \sin(\alpha_1 - \alpha)|$, then by (2.1) we have

$$\gamma + \sqrt{\gamma^2 + |z|^2 \sin^2 \theta} < |z| \sin \theta < 2 \sin \alpha_1 \cos \theta + r_1 \sin \theta.$$

Thus, the real part of every point in $W(zB)$ is strictly less than $2 \sin \alpha_1 \cos \theta + r_1 \sin \theta$, and not equal to λ . Since this is true for any $z \in W(A)$, we get the desired contradiction. \blacksquare

Proof of Theorem 1.2. We want to show that the implication **(A3)** \Rightarrow **(A1)** is valid when A is compact. Let $A \in B(H)$ be compact satisfying **(A3)**. Since A is compact, every nonzero element in $\sigma(A)$ is an eigenvalue of A . Hence we have $\sigma(A) \setminus \{0\} \subseteq W(A)$. In view of Lemma 2.3, we can assume that $\|A\| = r(A) = w(A) = 1$ which is an eigenvalue of A , and write $A = I \oplus A_1$ such that $\|A_1\| \leq 1$ and $1 \notin \sigma(A_1)$. Note that the largest eigenvalue λ of the compact operator $(A_1 + A_1^*)/2$ is less than 1. Indeed, if there is a unit vector x such that $(A_1 + A_1^*)x/2 = x$, then the inequality

$$1 = \left\langle \frac{A_1 + A_1^*}{2} x, x \right\rangle = \frac{1}{2} \langle A_1 x, x \rangle + \frac{1}{2} \langle A_1^* x, x \rangle \leq 1$$

implies that $1 = \langle A_1x, x \rangle$ and hence $A_1x = x$, contradicting that $1 \notin \sigma(A_1)$. Since

$$W(A_1) \subseteq \{\nu \in \mathbf{C} : |\nu| \leq 1, |1 - \nu| \leq 1, \text{ and } (\nu + \bar{\nu})/2 \leq \lambda\},$$

and $W(A) = \text{conv}(\{1\} \cup W(A_1))$, we see that there are two different support lines of $W(A)$ passing through 1. It then holds **(A1)** by Proposition 2.2. \blacksquare

Proof of Corollary 1.6. The assertions in (1) are clear. For (2), let $A \in B(H)$ be normal satisfying **(A3)** and let, without loss of generality, $1 = \|A\|$ be an isolated point in the spectrum $\sigma(A)$ of A . Write $A = I_1 \oplus A_1$ (an orthogonal sum), where I_1 is the eigen-projection of A for 1, and $1 \notin \sigma(A_1)$. Moreover, we can separate 1 from $\sigma(A_1)$ by a straight line in the complex plane. Consequently, 1 and $\overline{W(A_1)} = \text{conv } \sigma(A_1)$ are contained in two disjoint open half spaces. Since $W(A)$ is the convex hull of the set $\{1\} \cup W(A_1)$, we see that 1 lies on two different support lines of $\overline{W(A)}$. Proposition 2.2 applies and finishes the proof. \blacksquare

Verification of Example 1.3. By the Berberian construction, we may assume that $A = I + T$ such that T is normal and every point e^{it} on the unit circle is an eigenvalue. Suppose that $\lambda \in \sigma(AB)$. The case $\lambda = 0$ is done, since $0 \in W(A)W(B)$ as $0 \in W(A)$. Suppose $\lambda \neq 0$. Because $\sigma(AB)$ and $\sigma(BA)$ have the same nonzero elements, we see that $\lambda \in \sigma(BA)$.

Assume first that λ is an approximate eigenvalue of BA . By the Berberian construction, we may assume that there is a unit vector x such that $BAx = \lambda x$. Let $Ax = a_{11}x + a_{21}y$ such that $a_{11} = \langle Ax, x \rangle$ and y is a unit vector orthogonal to x . Because $A - I$ is unitary, we have

$$(2.4) \quad |a_{11} - 1|^2 + |a_{21}|^2 = 1.$$

Using an orthonormal basis with x, y as the first two vectors, and abusing notations for matrices of uncountable sizes, we see that the operator matrices of A and B have the form

$$\begin{pmatrix} a_{11} & * & * \\ a_{21} & * & * \\ 0 & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B_1 & * \\ * & * \end{pmatrix} \quad \text{with} \quad B_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then $W(B_1) \subseteq W(B)$,

$$\lambda = b_{11}a_{11} + b_{12}a_{21} \quad \text{and} \quad b_{21}a_{11} + b_{22}a_{21} = 0.$$

It follows from (2.4) that $a_{11} - 1$ lies in the closed unit complex disk. Hence we can write

$$a_{11} - 1 = -\alpha + (1 - \alpha)e^{ir}$$

for some $r \in [0, 2\pi) \setminus \{\pi\}$ and some $\alpha \in [0, 1]$.

Let $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 + e^{ir} \end{pmatrix}$ be the compression of A on the two dimensional subspace spanned by $\{f_\pi, f_r\}$. Let $u = \sqrt{\alpha}f_\pi + \sqrt{1 - \alpha}f_r$. Then

$$\langle (A_0 - I_2)u, u \rangle = a_{11} - 1.$$

Because $A_0 - I_2$ is unitary, $\|(A_0 - I_2)u\| = 1$. In view of (2.4), we see that $A_0 - I_2$ is unitarily similar to a matrix of the form $\begin{pmatrix} a_{11} - 1 & * \\ a_{21} & * \end{pmatrix}$. Hence, A_0 is unitarily similar to $A_1 = \begin{pmatrix} a_{11} & * \\ a_{21} & * \end{pmatrix}$ and

$W(A_1) = W(A_0) \subseteq W(A)$. Note that B_1A_1 is in upper triangular form with λ lying in the $(1, 1)$ position. Thus, $\lambda \in \sigma(B_1A_1)$. Note that A_0 , as well as A_1 , is a multiple of positive semi-definite matrix. By the implication **(A1)** \Rightarrow **(A2)** in Theorem 1.5, we have

$$\lambda \in \sigma(B_1A_1) \subseteq W(B_1)W(A_1) \subseteq \overline{W(B)W(A)}.$$

At this point, we have shown that $\overline{W(B)W(A)}$ contains all approximate eigenvalues of BA . Let $\alpha \in \sigma(BA)$ and α is not an approximate eigenvalue of BA . Then α is in the interior of $\sigma(BA)$ and thus there is an approximate eigenvalue λ of BA , which is a boundary point of $\sigma(BA)$, such that $\alpha = \beta\lambda$ with $0 < \beta < 1$. Since $\lambda \in \overline{W(B)W(A)} = \bigcup_{|z-1| \leq 1} z\overline{W(B)}$, we have $\alpha = \beta z b$ for some $b \in \overline{W(B)}$ and z satisfying $|z - 1| \leq 1$. Since $|\beta z - 1| \leq \beta|z - 1| + (1 - \beta) \leq 1$, we have $\alpha \in \overline{W(B)W(A)}$ as well. \blacksquare

Verification of Example 1.4. For any B in $B(H)$, we show that $\sigma(\hat{A}B) \subseteq \sigma(B\hat{A}) \cup \{0\} \subseteq W(\hat{A})W(B)$. Since $0 \in W(\hat{A})$, we have $0 \in W(\hat{A})W(B)$. So, we focus on those nonzero $\lambda \in \sigma(B\hat{A})$.

Similar to the Verification of Example 1.3, we only need to consider the case when λ is a nonzero approximate eigenvalue of $B\hat{A}$. Using a similar argument as in the Verification of Example 1.3, we may assume that the operator matrices of \hat{A} and B have the form

$$\begin{pmatrix} a_{11} & * & * \\ a_{21} & * & * \\ 0 & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B_1 & * \\ * & * \end{pmatrix} \quad \text{with} \quad B_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then $W(B_1) \subseteq W(B)$, and

$$\lambda = b_{11}a_{11} + b_{12}a_{21} \quad \text{and} \quad b_{21}a_{11} + b_{22}a_{21} = 0.$$

Because $\hat{A} - I$ is a contraction, we see that

$$|a_{11} - 1|^2 + |a_{21}|^2 \leq 1.$$

We can then construct a unitary matrix $A_0 \in M_3$ with first column equal to $(a_{11} - 1, a_{21}, a_{31})^t$, where $a_{31} = (1 - |a_{11} - 1|^2 - |a_{21}|^2)^{1/2}$. Since $A - I$ is a unitary operator with spectrum $\{e^{it} : t \in [0, 2\pi)\}$, we may regard A_0 as a compression of $A - I$, and hence $I + A_0$ is a compression of A and can be viewed as the leading principal submatrix of UAU^* , whose first column has only three nonzero entries, namely, a_{11}, a_{21}, a_{31} . So, the first column of $(B_1 \oplus [0])(I + A_0)$ equals $(\lambda, 0, 0)^t$, and thus the first column of $(B_1 \oplus O)(UAU^*)$ has only one nonzero entry λ lying in the $(1, 1)$ position. Since A satisfies **(A2)**, we have

$$\begin{aligned} \lambda &\in \sigma((B_1 \oplus O)(UAU^*)) = \sigma((U^*(B_1 \oplus O)U)A) \\ &\subseteq \sigma(A(U^*(B_1 \oplus O)U)) \cup \{0\} \subseteq W(A)W(U^*(B_1 \oplus O)U) = W(A)W(B_1 \oplus O). \end{aligned}$$

Note that $W(B_1 \oplus O) = \text{conv}\{W(B_1) \cup W(O)\} = \{rb : b \in W(B_1), r \in [0, 1]\}$. By the convexity of $W(A)$ and the fact that $0 \in W(A)$, if $a \in W(A)$ and $r \in [0, 1]$, then $ra \in W(A)$. Therefore, $\lambda \in W(A)W(B_1 \oplus O)$ implies that $\lambda = a(rb)$ with $a \in W(A)$, $b \in W(B_1) \subseteq W(B)$, $r \in [0, 1]$. It follows that $\lambda = (ra)b \in W(A)W(B) = W(\hat{A})W(B)$. \blacksquare

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